BMS flux balance equations

BMS symmetries and gravitational waves

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Based on arXiv:1912.03164 with *Geoffrey Compère* and *Roberto Oliveri*

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Motivation

Context: Gauge symmetries, redundancy or physical?
Textbooks: *Redundancy*
My answer: *Partly physical*

Related physical concepts

- Asymptotic symmetries, (GR, hep-th)
- adiabatic modes, Ward identities (cosmology)
- edge states (cond-mat)
- Berry phases and Aharanov Bohm effect in QM

Symmetries and conservation laws

Noether's Theorem [*E.Noether(1918)*]

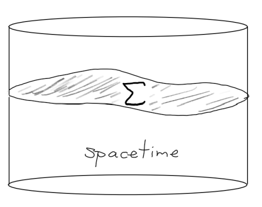
(simplified version) For a continuous symmetry $\phi \to \phi + \delta_\epsilon \phi$ preserving the Lagrangian $\delta_\epsilon \mathcal{L} = 0$, the following current is conserved

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_\epsilon \phi, \qquad \partial_\mu J^\mu_\epsilon = 0\,.$$

The *Noether charge*

$$Q_\epsilon = \int_\Sigma d^3x\, n_\mu\, J^\mu_\epsilon$$

is a constant of motion for closed systems.



Can we use this for gauge symmetries as well? Let's try an example.

Example: Maxwell theory *[AS '16]*

The Lagrangian and its symmetry

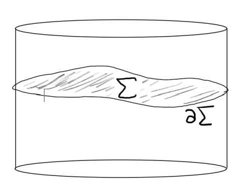
$$\mathcal{L}[A] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \delta_\lambda A_\mu = \partial_\mu \lambda(x)$$

Compute Noether current

$$\begin{aligned} J^\mu_\lambda &= F^{\mu\nu}\partial_\nu \lambda \\ &= \partial_\nu(\lambda F^{\mu\nu}) - \lambda \partial_\nu F^{\mu\nu} \approx \partial_\nu(\lambda F^{\mu\nu}) \end{aligned}$$

Noether charge

$$Q_\lambda = \int_\Sigma d^3x\, n_\mu\, J^\mu \approx \oint_{\partial\Sigma} dS_{\mu\nu} \lambda F^{\mu\nu}$$



Remark) Noether charge is a boundary integral.

Remark 2) $\lambda = 1$ gives total electric charge.

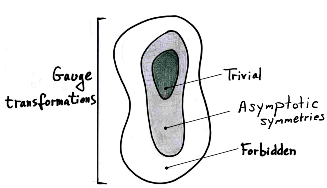
Asymptotic symmetries

General theorem *[Wald'93, Barnich,Brandt'01]*

The charge associated to local symmetries can always be written as boundary integrals.

Classification of gauge symmetries

- Pure gauge transformations $Q_\lambda = 0$
- Large gauge transformations or asymptotic symmetries $Q_\lambda \neq 0$
- $Q_\lambda = \infty$ forbidden by boundary conditions



Credit: *[Oblak'16]*

Adiabatic modes

Adiabatic modes *[Weinberg'05]*

Adiabatic modes are perturbations that in the long wavelength limit take the form of a gauge transformation

- Example: Maxwell theory in temporal gauge *[Mirbabayi,Simonović '16- AS,van den Bleeken '17]*
- It turns out that adiabatic modes are given by (see back up slides)

$$A_i = \partial_i(t\,\lambda(\boldsymbol{x})), \qquad \nabla^2\lambda(\boldsymbol{x}) = 0$$

- Remark. Regular harmonic functions cannot vanish at the boundary. Correspondence between adiabatic modes and asymptotic symmetries.

Hamiltonian approach

Hamiltonian approach provides a great framework to discuss symmetries and conservation laws [*Regge, Teitelboim'74*]

- Define algebra of charges
- Hidden symmetries
- More physical expression for charge $\rightarrow$ E.g. $1/2$ discrepancy betweem mass and angular momentum in Komar integrals

Covariant phase space [*Ashtekar '82, Wald '90, Barnich, Brandt '01*]

PART II:
Asymptotic symmetries in gravity

Asymptotic symmetries in gravity

- Local (gauge) symmetries of GR= Diffeomorphisms, arbitrary coordinate transformations $x^\mu \to \tilde{x}^\mu(x)$
- infinitesimal form $x^\mu \to x^\mu + \xi^\mu(x)$
- Asymptotic symmetries are those coordinate transformations that preserve the boundary conditions and are associated to finite charges
  - preserve the boundary conditions,
  - are associated to finite charges.

Asymptotic symmetries and holography

Commun. Math. Phys. 104, 207–226 (1986)

Communications in Mathematical Physics

Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three Dimensional Gravity

J. D. Brown and Marc Henneaux*,**

Center for Theoretical Physics, The University of Texas at Austin, Austin, Texas 78712, USA

Abstract. It is shown that the global charges of a gauge theory may yield a nontrivial central extension of the asymptotic symmetry algebra already at the classical level. This is done by studying three dimensional gravity with a negative cosmological constant. The asymptotic symmetry group in that case is either $R \times SO(2)$ or the pseudo-conformal group in two dimensions, depending on the boundary conditions adopted at spatial infinity. In the latter situation, a nontrivial central charge appears in the algebra of the canonical generators, which turns out to be just the Virasoro central charge.

(Asymptotic) symmetries of gravity AdS_3 = Symmetries of CFT in 2d

Asymptotically flat spacetime

Gravitational waves in general relativity
VII. Waves from axi-symmetric isolated systems

By H. Bondi, F.R.S., M. G. J. van der Burg and A. W. K. Metzner

(*Received 8 January* 1962—*Revised 2 April* 1962)

Gravitational waves in general relativity
VIII. Waves in asymptotically flat space-time

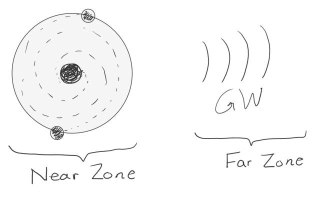
By R. K. Sachs*
King's College, University of London

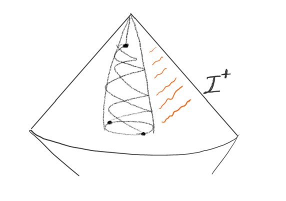
(*Communicated by H. Bondi, F.R.S.—Received 7 May* 1962)

Introduction of the "BMS" symmetries

Asymptotically flat spacetimes

A setup to study gravitational waves from compact sources





Structure of (linearized) wave solution $\quad \psi \sim \dfrac{f(t-r)}{r}$

Introduce the null coordinate retarded time $u = t - r + \mathcal{O}(G)$

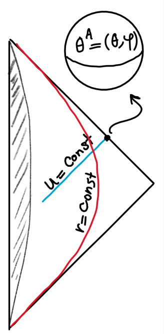
GR in Bondi gauge

Bondi gauge: Use retarded coordinates (u, r, θ^A) such that

$$g_{rr} = g_{rA} = 0, \qquad \partial_r \det\left(r^{-2} g_{AB}\right) = 0$$

$$ds^2 = -e^{2\beta}(U du^2 + 2du dr) + r^2 h_{AB}(d\theta^A - U^A du)(d\theta^B - U^B du)$$

Boundary conditions: $\lim_{r\to\infty}(g_{\mu\nu} - \eta_{\mu\nu}) = 0$



Einstein equations solve for (β, U, U^A, h_{AB}) [*Winicour '83*]

4 hypersurface equations

2 evolution equations

4 constraint equations

Bondi asymptotic expansion

Expansion in $1/r$ and imposing Einstein equations

$$
\begin{aligned}
ds^2 = &- c^2\, du^2 - 2c\, dudr + r^2 \gamma_{AB}\, d\theta^A d\theta^B \\
&+ \frac{2G}{c^2 r} m\, du^2 + \frac{1}{r}\frac{4G}{3c^2} N_A\, dud\theta^A + r\, C_{AB}\, d\theta^A d\theta^B \\
&+ \text{subleading in } 1/r
\end{aligned}
$$

Bondi data: $m(u,\theta^A), N_A(u,\theta^A)$ Bondi mass, angular momentum aspect, $C_{AB}(u,\theta^A)$ Bondi shear is traceless $(= \lim_{r\to\infty} h^{TT}_{ij})$

Remaining Einstein equations

$$
\begin{aligned}
\partial_u m &= -\frac{c^3}{8G}\dot{C}_{AB}\dot{C}^{AB} + \frac{c^4}{4G} D_A D_B \dot{C}^{AB}, \\
\partial_u N_A &= \partial_A m + \frac{c^3}{4G}\left(D_B(\dot{C}^{BC} C_{CA}) + 2 D_B \dot{C}^{BC} C_{CA}\right) \\
&\quad + \frac{c^4}{4G} D^B (D_A D^C C_{BC} - D_B D^C C_{AC}).
\end{aligned}
$$

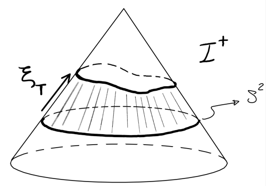
BMS symmetries

BMS symmetries

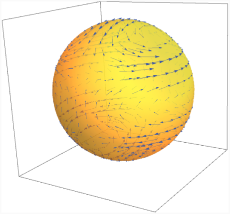
Definition Tansformations preserving the form of metric in Bondi gauge but change the Bondi data

Extended BMS generators

$$\xi = \underbrace{T(\theta)\partial_u}_{\text{supertranslation}} + \underbrace{Y^A(\theta)\partial_A - \frac{1}{2} r D_A Y^A \partial_r}_{\text{superLorentz}} + \text{subleading}$$



(a) Supertranslations



(b) superLorentz

They form an infinite dimensional Lie algebra

BMS symmetries

A vector field on sphere can be expanded as

$$Y^A = \epsilon^{AB}\partial_A \Phi \partial_B + \gamma^{AB}\partial_A \Psi \partial_B$$

$$\xi = \underbrace{T(\theta^A)\partial_u}_{\text{supertranslation}} + \underbrace{\epsilon^{AB}\partial_B \Phi\, \partial_A}_{\text{superrotation}} + \underbrace{\partial^A \Psi\, \partial_A - \frac{1}{2} r D^2 \Psi\, \partial_r}_{\text{superboost}}$$

Poincare subalgebra

	T	Φ	Ψ
$Y_{0,0}$	time translation	-	-
$Y_{1,m}$	spatial translation	rotations	boosts
$Y_{\ell,m}$	supertranslation	superrotations	superboosts

BMS charges

BMS charges [*Campiglia, Laddha '17, Compère et.al. '18*]

$$\mathcal{P}_T = \frac{1}{c} \oint_S T\, m\,, \qquad \textit{Supermomentum}$$

$$\mathcal{J}_\Phi = \frac{1}{2} \oint_S \epsilon^{AB} \partial_B \Phi N_A, \qquad \textit{S-angular momentum}$$

$$\mathcal{K}_\Psi = \frac{1}{2c} \oint_S \gamma^{AB} \partial_B \Psi N_A \qquad \textit{S-center of mass}$$

Charge pairs the symmetry parameter with the Bondi aspect

BMS balance equations

BMS charges are NOT conserved charges. Instead they obey certain balance equations

BMS balance equations

Balance equations *[Peters, Mathews '63]*

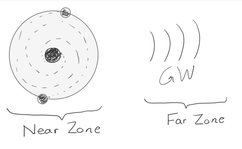
No radiation: Conservation laws enough to solve Kepler problem

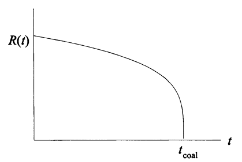
Radiation: Balance equations replace conservation laws

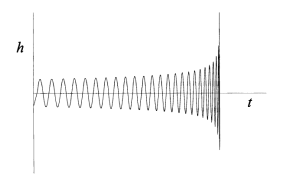
$$\frac{dE_{mech}}{dt} = -\mathcal{F}_{rad}$$

To lowest order approximation

$$E_{mech} = -\frac{Gm_1 m_2}{2r}, \qquad \mathcal{F}_{rad} = \frac{G}{5c^5} \dddot{I}_{ij} \dddot{I}_{ij}$$







Find balance equations for BMS charges

BMS flux balance equations

Remind the equation for Bondi mass aspect

$$\dot{m} = -\frac{c^3}{8G}\dot{C}_{AB}\dot{C}^{AB} + \frac{c^4}{4G}D_A D_B \dot{C}^{AB}$$

Flux balance equation for supermomentum $\mathcal{P}_T = \frac{1}{c}\oint_S T\, m$

$$\boxed{\dot{P}_T = \frac{1}{c}\oint_S T\left(-\frac{c^3}{8G}\dot{C}_{AB}\dot{C}^{AB} + \frac{c^4}{4G}D_A D_B \dot{C}^{AB}\right)}$$

Similarly for superLorentz charges

Multipole expansion

Multipole expansion

- Why multipole expansion? PN hierarchy of fluxes
- Multipole expansion of radiation field $\left(n_i = \frac{x_i}{r}, N_L = n_{i_1} \cdots n_{i_\ell}\right)$

$$C_{ij} = \sum_{\ell=2}^{+\infty} \frac{4G}{c^{\ell+2}\ell!} \left(N_{L-2}\, \boldsymbol{U}_{ijL-2} - \frac{b_\ell}{c} N_{aL-2}\, \epsilon_{ab(i} \boldsymbol{V}_{j)bL-2} \right)^{TT}$$

in terms of mass and spin radiative multipole moments.

- STF Multipole expansion of symmetries

$$T(\theta^A) = \sum_{\ell=0}^{\infty} \boldsymbol{T}_L N_L\,, \quad \Phi(\theta^A) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \boldsymbol{S}_L N_L\,, \quad \Psi(\theta^A) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \boldsymbol{K}_L N_L$$

- One balance equation for each harmonic of the symmetry parameters

Poincarè flux balance equations

$$\dot{P}_T = \frac{1}{c}\oint_S T\left(-\frac{c^3}{8G}\dot{C}_{AB}\dot{C}^{AB} + \frac{c^4}{4G}D_A D_B \dot{C}^{AB}\right)$$

Energy and angular momentum [*Blanchet, Faye '18*]

$$\dot{\mathcal{E}} = -\sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+1}}\mu_\ell\left\{\dot{\boldsymbol{U}}_L\dot{\boldsymbol{U}}_L + \frac{b_\ell b_\ell}{c^2}\dot{\boldsymbol{V}}_L\dot{\boldsymbol{V}}_L\right\},$$

$$\dot{\mathcal{J}}_i = -\varepsilon_{ijk}\sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+1}}\ell\mu_\ell\left\{\boldsymbol{U}_{jL-1}\dot{\boldsymbol{U}}_{kL-1} + \frac{b_\ell b_\ell}{c^2}\boldsymbol{V}_{jL-1}\dot{\boldsymbol{V}}_{kL-1}\right\},$$

$$\dot{\mathcal{P}}_i = -\sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+3}}\left\{2(\ell+1)\mu_{\ell+1}\left(\dot{\boldsymbol{U}}_{iL}\dot{\boldsymbol{U}}_L + \frac{b_\ell b_{\ell+1}}{c^2}\dot{\boldsymbol{V}}_{iL}\dot{\boldsymbol{V}}_L\right) + \sigma_\ell\,\varepsilon_{ijk}\dot{\boldsymbol{U}}_{jL-1}\dot{\boldsymbol{V}}_{kL-1}\right\}$$

where $\quad \mu_\ell = \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}, \quad \sigma_\ell = \frac{8(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}, \quad b_\ell = \frac{\ell}{\ell+1}$

Similar results for center of mass.

Leading terms

Radiative vs. source multipoles

$$\boldsymbol{U}_L = I_L^{(\ell)} + \mathcal{O}\left(\frac{G}{c^3}\right), \qquad \boldsymbol{V}_L = J_L^{(\ell)} + \mathcal{O}\left(\frac{G}{c^3}\right)$$

Leading terms

$$\dot{\mathcal{E}} = -\frac{G}{c^5}\left(\frac{1}{5}I_{ij}^{(3)}I_{ij}^{(3)}\right) - \frac{G}{c^7}\left(\frac{1}{189}I_{ijk}^{(4)}I_{ijk}^{(4)} + \frac{16}{45}J_{ij}^{(3)}J_{ij}^{(3)}\right) + O(c^{-9})$$

$$\dot{\mathcal{J}}_i = -\frac{G}{c^5}\left(\frac{2}{5}\epsilon_{ijk}I_{jl}^{(2)}I_{kl}^{(3)}\right) - \frac{G}{c^7}\epsilon_{ijk}\left(\frac{1}{63}I_{jlm}^{(3)}I_{klm}^{(4)} + \frac{32}{45}J_{jl}^{(2)}J_{kl}^{(3)}\right) + O(c^{-9})$$

$$\dot{\mathcal{P}}_i = -\frac{G}{c^7}\left(\frac{2}{63}I_{ijk}^{(4)}I_{jk}^{(3)} + \frac{16}{45}\epsilon_{ijk}I_{jl}^{(3)}J_{kl}^{(3)}\right) + O\left(c^{-9}\right).$$

$$\dot{\mathcal{G}}_i = \mathcal{P}_i - \frac{G}{c^7}\left[\frac{1}{21}\left(I_{jk}^{(3)}I_{ijk}^{(3)} - I_{jk}^{(2)}I_{ijk}^{(4)}\right)\right] + O\left(c^{-9}\right).$$

Matches exactly with standard results[*Einstein '18, Epstein, Wagoner '75, Thorne '80, Kozameh '16, Blanchet, Faye '18*]

Supermomentum flux

General structure

$$\dot{\mathcal{P}}_L = \left[\dot{\mathcal{P}}_L\right]_{\text{lin}} + \left[\dot{\mathcal{P}}_L\right]^{+}_{\text{quad}} + \left[\dot{\mathcal{P}}_L\right]^{-}_{\text{quad}}.$$

Linear flux

$$\left[\dot{\mathcal{P}}_L\right]_{\text{lin}} = \frac{\Theta_{\ell-2}}{c^{\ell-1}} \frac{(\ell+2)(\ell+1)}{2(2\ell+1)!!} \boldsymbol{T}_L \dot{\boldsymbol{U}}_L,$$

Quadratic flux-parity even

$$\left[\dot{\mathcal{P}}_L\right]^{+}_{\text{quad}} = -\sum_{\ell',\ell''=2}^{\infty} \frac{G\,\mu_{\ell\ell'\ell''}}{c^{\ell''+\ell'+2}} \boldsymbol{T}_{L_1L_2} \Big(\dot{\boldsymbol{U}}_{L_1L_3} \dot{\boldsymbol{U}}_{L_2L_3} + \frac{b_{\ell'} b_{\ell''}}{c^2} \dot{\boldsymbol{V}}_{L_1L_3} \dot{\boldsymbol{V}}_{L_2L_3} \Big) \delta_{\ell'',\ell',\ell}.$$

Leading terms

$$\dot{\mathcal{P}}_{ij} - \frac{2}{5c}\dot{U}_{ij} = +\frac{G}{c^6}\left[\frac{4}{35}\left(\dot{U}_{ik}\dot{U}_{jk} - \frac{1}{3}\delta_{ij}\dot{U}_{kl}\dot{U}_{kl}\right)\right] + \mathcal{O}(c^{-8}),$$

$$\dot{\mathcal{P}}_{ijkl} - \frac{1}{63c^3}\dot{U}_{ijkl} = -\frac{G}{c^6}\left(\frac{2}{315}\dot{U}_{\langle ij}\dot{U}_{kl\rangle}\right) + \mathcal{O}(c^{-8})$$

PN analysis of BMS fluxes

PN order of quadratic fluxes

Supermomentum	
ℓ-pole	PN order
0, 2, 4	3
1, 3, 5	3.5

s-ang.momemtum	
ℓ-pole	PN order
1, 3	2.5
2, 4	3

s-center of mass	
ℓ-pole	PN order
1, 3, 5	3.5
2, 4, 6	4

Example: Octupolar super angular momentum

$$\dot{\mathcal{J}}_{ijk} - \frac{2}{7c^2}\, u\, \dot{\boldsymbol{V}}_{ijk} = -\frac{G}{c^5}\left(\frac{6}{35}\,\epsilon_{pq\langle i}\,\dot{\boldsymbol{U}}_{j|p|}\boldsymbol{U}_{k\rangle q}\right) + \mathcal{O}(c^{-7})$$

Conclusion

Summary

- New flux balance equations
- At the same PN order as that of energy and angular momentum
- In principle relevant for radiation reaction forces

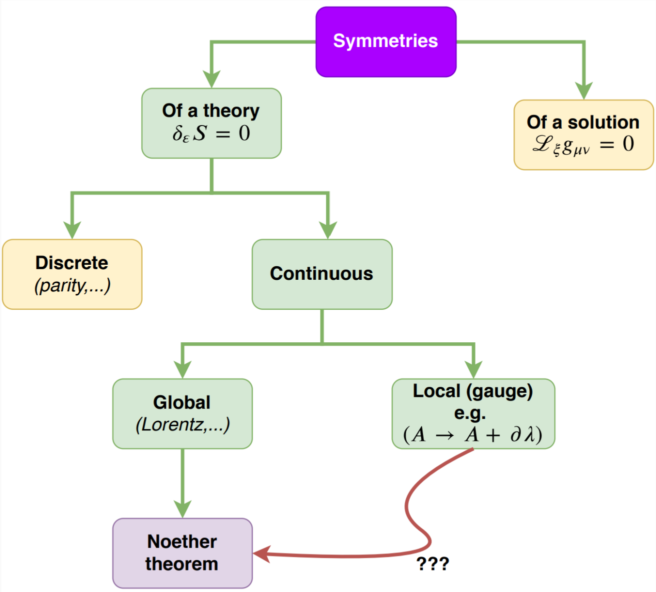
Outlook

- Wrtie BMS charges in terms of source variables
- New differential equations for source parameters

Thank you very much

Back up

Noether's theorem



Adiabatic modes

Adiabatic modes *[Weinberg'05]*

Adiabatic modes are perturbations that in the long wavelength limit take the form of a gauge transformation

- Example: Maxwell theory in temporal gauge *[Mirbabayi,Simonović '16- AS,van den Bleeken '17]*

$$A_0 = 0 \implies A_i \to A_i + \partial_i \lambda(\boldsymbol{x})$$

- Gauss equation

$$\nabla \cdot E = \partial_i \dot{A}_i = 0 \qquad \text{trivially solved by} \qquad A_i = \partial_i \lambda(\boldsymbol{x})$$

- Introducing slow time dependence $A_i = \partial_i \lambda(\epsilon t, \boldsymbol{x})$ and requiring e.o.m= $\mathcal{O}(\epsilon^2)$ implies

$$A_i = \partial_i(\epsilon t\, \lambda(\boldsymbol{x})), \qquad \nabla^2 \lambda(\boldsymbol{x}) = 0$$

- Remark. Regular solutions cannot vanish at the boundary. Correspondence between adiabatic modes and asymptotic symmetries.