Classical scattering of spinning black holes from quantum amplitudes

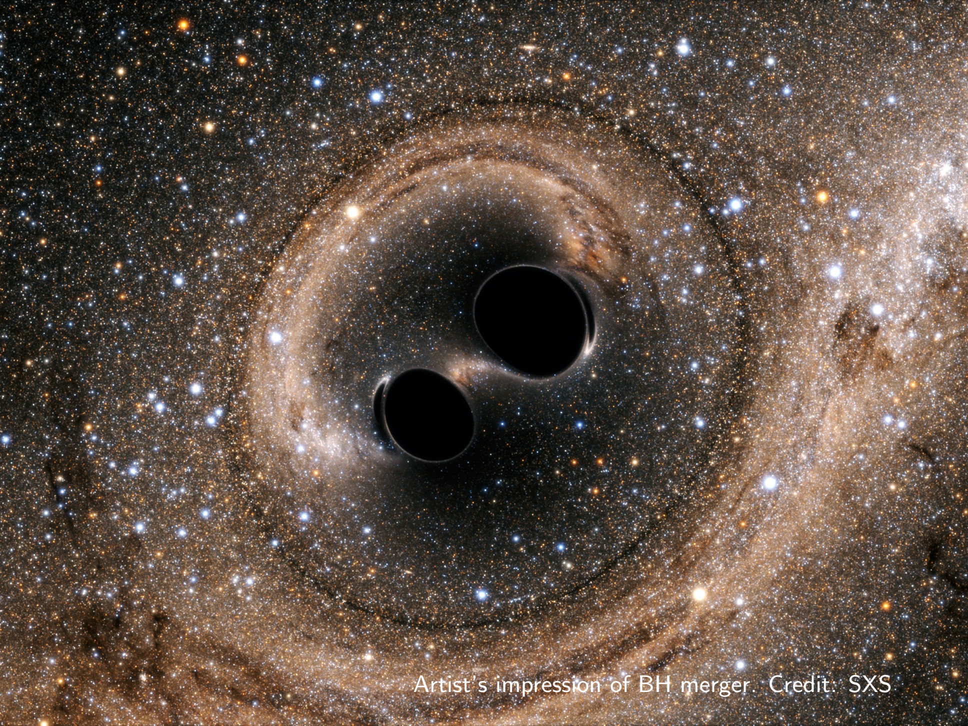
based on work with Alfredo Guevara and Justin Vines
arXiv:1812.06895, 1906.10071 [hep-th]

Alexander Ochirov
ETH Zürich

GReCO seminar,
Institut d'Astrophysique de Paris, Dec 16, 2019

Introduction

“photo” by Event Horizon Telescope Collaboration ’19

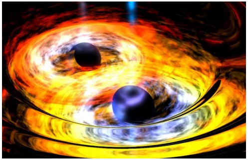


Artist's impression of BH merger. Credit: SXS

Conferences and Workshops

QCD Meets Gravity Workshop 2019

December 9-13, 2019



This is the fifth in a series of meetings for researchers interested in the remarkable correspondence between Yang-Mills theory and gravity, known as the double-copy construction. An early manifestation of this is the KLT relations. The BCJ double-copy construction has greatly simplified multiloop perturbative computations in gravity theories, leading to new insight into the ultraviolet properties of gravity theories. Currently there is an intense global research activity to understand the origin of the relation as well as to apply it to more general classical solutions in General Relativity, including the important problem of gravitational radiation from compact astrophysical objects. This meeting brings together experts in both gauge and gravity theories including supersymmetric extensions, and focus bringing the methodologies developed for gauge theories to handle problems in gravity.

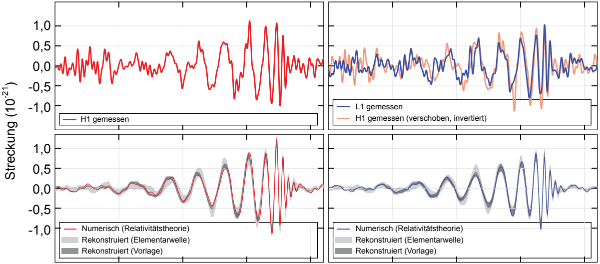
For more information regarding the workshop, please click here.

Previous workshops in this series were:

- QCD Meets Gravity I, Higgs Centre, Edinburgh, April 2016.
- QCD Meets Gravity II, Bhaumik Institute, UCLA, December 2016.
- QCD Meets Gravity III, Bhaumik Institute, UCLA, December 2017.
- QCD Meets Gravity IV, Nordita, December 2018.

Motivation by data

- BH merger GW150914 seen by LIGO+Virgo



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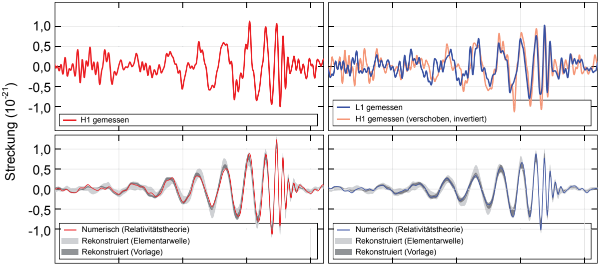
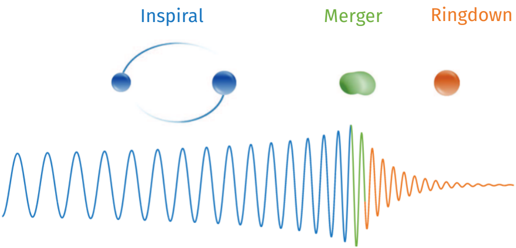


- General case:

picture from Antelis, Moreno '16



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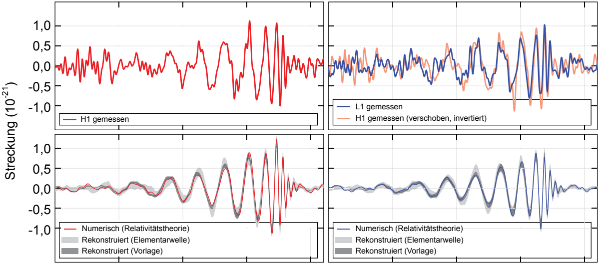

- Early discovery searches: rough waveform templates, or none
- Measurement mode: precise & fast templates to cover param. space

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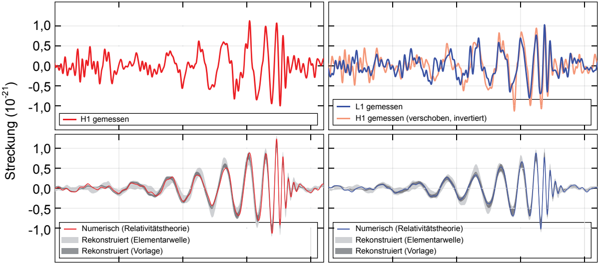

- Early discovery searches: rough waveform templates, or none
- Measurement mode: precise & fast templates to cover param. space
- NR excellent but expensive/slow, sensible for merger phase
- Inspiral phase suitable for 2-body Post-Newtonian theory
- Exp. error will decrease in adv. LIGO, LISA, etc.
- Length of inspiral signal will grow $\Rightarrow$ need for more pert. results

Analytic perturbation schemes

Limit	Perturbation theory	Natural for
Newtonian gravity $c \to \infty$	post-Newtonian $\frac{m_1}{m_2} \sim 1, \qquad \frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1$	bound orbits
special relativity $G \to 0$	post-Minkowskian $\frac{m_1}{m_2} \sim 1, \qquad \frac{Gm}{rc^2} \ll \frac{v^2}{c^2} \sim 1$	scattering
test-body motion in a stationary background $\frac{m_1}{m_2} \to 0$	post-test-body ("self-force") $\frac{m_1}{m_2} \ll 1, \qquad \frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \sim 1$	both

table by Vines

Motivation by peers

- Impressive progress by PN theory

4PN dynamics by: Damour, Jaranowski, Schäfer '14
Bernard, Blanchet, Bohé, Faye, Marchand, Marsat '15
Foffa, Mastrolia, Sturani, Sturm '16
Foffa, Porto, Rothstein, Sturani '19
5PN static by: Foffa, Mastrolia, Sturani, Sturm, Torres '19
Blümlein, Maier, Marquard '19

- EOB Hamiltonian from PM scattering instead of from PN 2-body bound-state dynamics

Buonanno, Damour '98 $\rightarrow$ Damour '16

- On-shell amplitude methods:
quantum gravity scattering easier than GR dynamics

e.g. 3PM 0-spin Hamiltonian by Bern, Cheung, Roiban, Shen, Solon, Zeng '19

- Classical $\leftarrow$ quantum relationship being sharpened

Guevara '17
Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18
Cheung, Rothstein, Solon '18
Kosower, Maybee, O'Connell '18
Koemans Collado, Di Vecchia, Russo '19
Bjerrum-Bohr, Cristofoli, Damgaard, Vanhove '19
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Damgaard, Haddad, Helset '19
Kälin, Porto '19

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This talk:

- **1PM** and 2PM BH scattering with **spin** from amplitudes

Outline

1. On-shell amplitudes
2. Spin exponentiation from minimal coupling
3. 1PM with general spin dependence
4. Aligned-spin results at 2PM
5. Summary & outlook

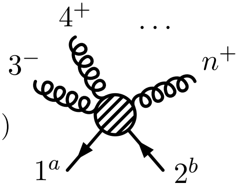
On-shell amplitudes

Why spinor helicity?

Consider QFT amplitude $\mathcal{A}(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)$*

Feynman rules give function of

- momenta p_i^μ
- polarization vectors $\varepsilon_\pm^\mu(p_i)$, tensors $\varepsilon_\pm^{\mu\nu}(p_i)$
- external spinors $\bar{v}^a(p_1)$, $u^b(p_2)$



*Disclaimer: all momenta incoming

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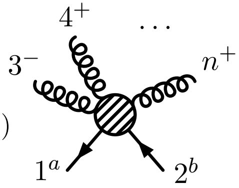

But all vector, spinor indices must be contracted

Remaining indices $\Leftrightarrow$ physical quantum numbers:

- helicities $\pm$ $\Leftrightarrow$ spins $\{\pm 1/2\}_p$, $\{\pm 1\}_p$, etc.
- $\mathrm{SU}(2)$ labels a, b $\Leftrightarrow$ spins $\{\pm 1/2\}_q$, $\{\pm 1, 0\}_q$, etc.

Crucial on-shell notion — LITTLE GROUP

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*Disclaimer: all momenta incoming

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3)$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP
  - massless states $\Leftarrow$ $\mathrm{SO}(2)$
  - massive states $\Leftarrow$ $\mathrm{SO}(3)$

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3) \quad \subset \quad \mathrm{SL}(2,\mathbb{C})$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP's dbl cover
  - massless states $\Leftarrow$ $\mathrm{SO}(2) \quad \subset \quad \mathbf{U(1)}$
  - massive states $\Leftarrow$ $\mathrm{SO}(3) \quad \subset \quad \mathbf{SU(2)}$

Minor complication: spinorial reps use groups' double covers

$\mathrm{U}(1)$ and $\mathrm{SU}(2)$ arise naturally in spinor helicity

Spinor map

Basis for spinor helicity

- Minkowski space isomorphism:*

$$\mathrm{M}^{2\times 2,\mathbb{C}}_{\mathrm{Hermitian}} \quad \leftrightarrow \quad \mathbb{R}^{1,3}$$

$$p_{\alpha\dot{\beta}} = p_\mu \sigma^\mu_{\alpha\dot{\beta}} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}$$

$$\det\{p_{\alpha\dot{\beta}}\} \; = \; m^2$$

*$\sigma^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

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$$\det\{p_{\alpha\dot\beta}\} = m^2$$

- Lorentz group homomorphism:

$$\mathrm{SL}(2,\mathbb{C}) \quad \to \quad \mathrm{SO}(1,3)$$
$$p_{\alpha\dot\delta} \to S_\alpha{}^\beta p_{\beta\dot\gamma} \left(S_\delta{}^\gamma\right)^* \quad \Rightarrow \quad p^\mu \to L^\mu{}_\nu p^\nu, \quad L^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\left(\bar\sigma^\mu S \sigma_\nu S^\dagger\right)$$

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Massless vs massive spinor helicity

Arkani-Hamed, Huang, Huang '17

MASSLESS	MASSIVE
$\det\{p_{\alpha\dot\beta}\} = 0$	$\det\{p_{\alpha\dot\beta}\} = m^2$
$p_{\alpha\dot\beta} = \lambda_{p\alpha}\tilde\lambda_{p\dot\beta} \equiv \|p\rangle_\alpha [p\|_{\dot\beta}$	$p_{\alpha\dot\beta} = \lambda_{p\alpha}^{\ a}\epsilon_{ab}\tilde\lambda_{p\dot\beta}^{\ b} \equiv \|p^a\rangle_\alpha [p_a\|_{\dot\beta}$ $\det\{\lambda_{p\alpha}^{\ a}\} = \det\{\tilde\lambda_{p\dot\alpha}^{\ a}\} = m$
$p^\mu = \frac{1}{2}\langle p\|\sigma^\mu\|p]$	$p^\mu = \frac{1}{2}\langle p^a\|\sigma^\mu\|p_a]$
$p_{\alpha\dot\beta}\tilde\lambda_p^{\dot\beta} = 0$	$p_{\alpha\dot\beta}\tilde\lambda_p^{a\dot\beta} = m\lambda_{p\alpha}^{\ a}$
$\langle p\,q\rangle = -\langle q\,p\rangle \quad \Rightarrow \quad \langle p\,p\rangle = 0$ $[p\,q] = -[q\,p] \quad \Rightarrow \quad [p\,p] = 0$ $\langle p\,q\rangle[q\,p] = 2p{\cdot}q$	$\langle p^a q^b\rangle = -\langle q^b p^a\rangle$ e.g. $\langle p^a p^b\rangle = -m\epsilon^{ab}$ $[p^a q^b] = -[q^b p^a]$ e.g. $[p^a p^b] = m\epsilon^{ab}$ $\langle p^a q^b\rangle[q_b\,p_a] = 2p{\cdot}q$

- can be used for on-shell BCFW recursion — AO '18

Little group transformations

Consider Lorentz transform $p^\mu \to L^\mu{}_\nu p^\nu \quad \leftrightarrow \quad \underset{\in SO(1,3)}{L^\mu{}_\nu} = \frac{1}{2}\,\mathrm{tr}\big(\bar{\sigma}^\mu \underset{\in SL(2,\mathbb{C})}{S} \sigma_\nu S^\dagger\big)$

MASSLESS:

$$|p\rangle \to S|p\rangle = e^{i\phi/2}|Lp\rangle \qquad \langle p| \to \langle p|S^{-1} = e^{i\phi/2}\langle Lp|$$

$$|p] \to S^{\dagger\,-1}|p] = e^{-i\phi/2}|Lp] \qquad [p| \to [p|S^\dagger = e^{-i\phi/2}[Lp|$$

$e^{ih\phi} \in \mathrm{U}(1)$ encode $2d$ rotations in frame where $p = (E, 0, 0, E)$

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$e^{ih\phi} \in \mathrm{U}(1)$ encode $2d$ rotations in frame where $p = (E, 0, 0, E)$

MASSIVE:

$$|p^a\rangle \to S|p^a\rangle = \omega^a{}_b|Lp^b\rangle \qquad |p^a\rangle \to |p^a\rangle S^{-1} = \omega^a{}_b|Lp^a\rangle$$

$$|p^a] \to S^{\dagger\,-1}|p^a] = \omega^a{}_b[Lp^b| \qquad [p^a| \to [p^a|S^\dagger = \omega^a{}_b[Lp^b|$$

$\omega \in \mathrm{SU}(2)$ encode $3d$ rotations in rest frame where $p = (m, 0, 0, 0)$

Wavefunctions from helicity spinors

Massless:

$$\varepsilon^{\mu}_{p+} = \frac{\langle q|\sigma^{\mu}|p]}{\sqrt{2}\langle q\,p\rangle} \qquad \varepsilon^{\mu}_{p-} = \frac{\langle p|\sigma^{\mu}|q]}{\sqrt{2}[p\,q]} \qquad \Rightarrow \qquad \begin{cases} \varepsilon^{\pm}_{p}\cdot p = \varepsilon^{\pm}_{p}\cdot q = 0 \\ \varepsilon^{\mu}_{p+}\varepsilon^{\nu}_{p-} + \varepsilon^{\mu}_{p-}\varepsilon^{\nu}_{p+} = -\eta^{\mu\nu} + \dfrac{p^{\mu}q^{\nu} + q^{\mu}p^{\nu}}{p\cdot q} \\ \varepsilon^{h_1}_{p}\cdot\varepsilon^{h_2}_{p} = -\delta^{h_1(-h_2)} \end{cases}$$

Xu, Zhang, Chang '85

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Massive:

$$\varepsilon^{ab}_{p\mu} = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m} \quad \Rightarrow \quad \begin{cases} p\cdot\varepsilon^{ab}_p = 0 \\ \varepsilon^{ab}_{p\mu}\varepsilon_{p\nu ab} = -\eta_{\mu\nu} + \dfrac{p_\mu p_\nu}{m^2} \\ \varepsilon^{ab}_p\cdot\varepsilon_{pcd} = -\delta^{(a}_{(c}\delta^{b)}_{d)} \end{cases}$$

Guevara, AO, Vines '18
Chung, Huang, Kim, Lee '18

and (symmetrized) tensor products thereof

Helicity basis

Arkani-Hamed, Huang, Huang '17

Take $p^\mu = (E, P\cos\varphi\sin\theta, P\sin\varphi\sin\theta, P\cos\theta)$

$$|p^a\rangle = \lambda_{p\alpha}^{\ a} = \begin{pmatrix} \sqrt{E-P}\cos\frac{\theta}{2} & -\sqrt{E+P}e^{-i\varphi}\sin\frac{\theta}{2} \\ \sqrt{E-P}e^{i\varphi}\sin\frac{\theta}{2} & \sqrt{E+P}\cos\frac{\theta}{2} \end{pmatrix}$$
$$[p^a| = \tilde{\lambda}_{p\dot{\alpha}}^{\ a} = \begin{pmatrix} -\sqrt{E+P}e^{i\varphi}\sin\frac{\theta}{2} & -\sqrt{E-P}\cos\frac{\theta}{2} \\ \sqrt{E+P}\cos\frac{\theta}{2} & -\sqrt{E-P}e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

Then

$$s^\mu(u_p^a) = \frac{1}{2m}\bar{u}_{pa}\gamma^\mu\gamma^5 u_p^a = (-1)^{a-1}s_p^\mu$$
$$s_p^\mu = \frac{1}{m}(P, E\cos\varphi\sin\theta, E\sin\varphi\sin\theta, E\cos\theta)$$

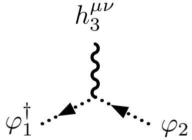
3-pt gravitational vertices

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \Rightarrow \qquad g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} + \mathcal{O}(\kappa^2)$$

Spin 0:

$$\mathcal{L}_{\text{scalar}} = g^{\mu\nu}(\partial_\mu\varphi)^\dagger(\partial_\nu\varphi) - m^2\varphi^\dagger\varphi$$
$$\mathcal{L}_{\varphi\varphi h} = -\kappa h^{\mu\nu}(\partial_\mu\varphi^\dagger)(\partial_\nu\varphi)$$

$\Rightarrow$



$\simeq i\kappa p_1^{(\mu} p_2^{\nu)}$

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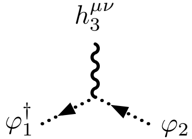
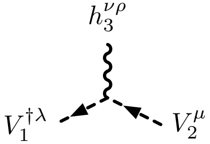


$$\simeq i\kappa p_1^{(\mu} p_2^{\nu)}$$

Spin 1:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{2} V^\dagger_{\mu\nu} V^{\mu\nu} + m^2 V^\dagger_\mu V^\mu$$
$$\mathcal{L}_{VVh} = \kappa h^{\mu\nu}\left(V^\dagger_{\mu\sigma} V_\nu{}^\sigma - m^2 V^\dagger_\mu V_\nu\right)$$

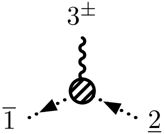
$\Rightarrow$



$$\simeq -i\kappa\Big[\big((p_1 \cdot p_2) + m^2\big)\eta^{\lambda(\nu}\eta^{\rho)\mu} + \eta^{\lambda\mu} p_1^{(\nu} p_2^{\rho)} - \eta^{\lambda(\nu} p_2^{\rho)} p_1^\mu - p_2^\lambda p_1^{(\nu} \eta^{\rho)\mu}\Big]$$

3-pt gravitational amplitudes

Spin 0:



$$= -i\kappa(p_1 \cdot \varepsilon_3)^2 = -\frac{i\kappa}{2} m^2 x_{\pm}^2$$

where $\quad x = \sqrt{2}\,\frac{p_1 \cdot \varepsilon_3}{m}$

3-pt gravitational amplitudes

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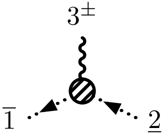

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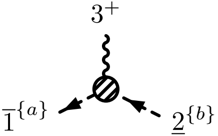
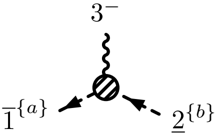


$$= -\frac{i\kappa}{2} x_{+}^2 \langle 1^{(a_1}\, 2^{(b_1} \rangle \langle 1^{a_2)}\, \overline{2}^{b_2)} \rangle$$



$$= -\frac{i\kappa}{2} x_{-}^2 [1^{(a_1}\, 2^{(b_1}] [1^{a_2)}\, \overline{2}^{b_2)}]$$

Minimal 3-pt amplitudes

Arkani-Hamed, Huang, Huang '17

$$\mathcal{M}_3(\overline{1}^{\{a\}}, \underline{2}^{\{b\}}, 3^+) = -\frac{\kappa}{2}\frac{\langle 1^a 2^b\rangle^{\odot 2s}}{m^{2s-2}} x_+^2$$

$$\mathcal{M}_3(\overline{1}^{\{a\}}, \underline{2}^{\{b\}}, 3^-) = -\frac{\kappa}{2}\frac{[1^a 2^b]^{\odot 2s}}{m^{2s-2}} x_-^2$$

$$x = \sqrt{2}\,\frac{p_1 \cdot \varepsilon_3}{m} : \qquad x_+ = \frac{\langle r|1|3]}{m\langle r\,3\rangle}, \qquad x_- = -\frac{[r|1|3\rangle}{m[r\,3]} = -\frac{1}{x_+}$$

NB! Independent of ref. momentum r

$$p_2^2 - m^2 = 2p_1 \cdot p_3 = \langle 3|1|3] = 0 \qquad \Rightarrow \qquad \exists\, x \in \mathbb{C} : |1|3\rangle = -mx|3]$$

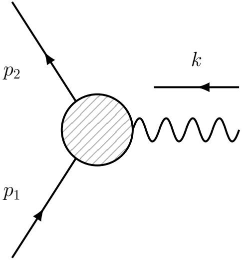
Spin exponentiation from minimal coupling

Spin exponentiation from minimal coupling

Want: extract classical spin dependence ($S^\mu \in \mathbb{R}^4$)
from quantum spin amplitudes ($s \in \mathbb{Z}_+$)

Minimal-coupling 3-pt amplitudes

Arkani-Hamed, Huang, Huang '17



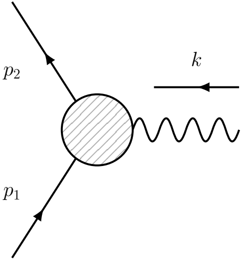
$$\mathcal{M}_3^{(s,+)} = -\frac{\kappa}{2}\frac{\langle 12\rangle^{\odot 2s}}{m^{2s-2}}x^2, \qquad x = -\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^+)$$

$$\mathcal{M}_3^{(s,-)} = -\frac{\kappa}{2}\frac{[12]^{\odot 2s}}{m^{2s-2}}x^{-2}, \qquad = \left[\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^-)\right]^{-1}$$

e.g. $\mathcal{M}_3^{(0,\pm)} = -\kappa(p_1\cdot\varepsilon^\pm)^2$

Minimal-coupling 3-pt amplitudes

Arkani-Hamed, Huang, Huang '17



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e.g. $\mathcal{M}_3^{(0,\pm)} = -\kappa(p_1\cdot\varepsilon^\pm)^2$

Angular-momentum structure inside:

$$\mathcal{M}_3^{(s,+)} = \mathcal{M}_3^{(0,+)}\frac{\langle 12\rangle^{\odot 2s}}{m^{2s}} = \frac{\mathcal{M}_3^{(0,+)}}{m^{2s}}[2|^{\odot 2s}\exp\left(-i\frac{k_\mu\varepsilon_\nu^+\bar{\sigma}^{\mu\nu}}{p_1\cdot\varepsilon^+}\right)|1]^{\odot 2s}$$

$$\mathcal{M}_3^{(s,-)} = \mathcal{M}_3^{(0,-)}\frac{[12]^{\odot 2s}}{m^{2s}} = \frac{\mathcal{M}_3^{(0,-)}}{m^{2s}}\langle 2|^{\odot 2s}\exp\left(-i\frac{k_\mu\varepsilon_\nu^-\sigma^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)|1\rangle^{\odot 2s}$$

Guevara, AO, Vines '18

inspired by soft theorems, e.g. Cachazo, Strominger '14

Angular-momentum exponential of Kerr

Vines '17

Stress-energy tensor (eff. source) for lin. Kerr BH:*

$$T^{\mu\nu}_{\mathsf{BH}}(x) = \frac{1}{m}\int d\tau\, p^{(\mu} \exp(a * \partial)^{\nu)}{}_{\rho} p^{\rho} \delta^{(4)}(x - u\tau), \qquad p^{\mu} = m u^{\mu}$$

$$T^{\mu\nu}_{\mathsf{BH}}(k) = \hat{\delta}(p \cdot k) p^{(\mu} \exp(-i a * k)^{\nu)}{}_{\rho}\, p^{\rho}, \qquad S^{\mu} = m a^{\mu}$$

*Hat notation absorbs straightforward powers of 2π.

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Couple to on-shell graviton $h_{\mu\nu}(k) \to \hat{\delta}(k^2)\varepsilon_{\mu}\varepsilon_{\nu}$:

$$h_{\mu\nu}(k) T^{\mu\nu}_{\mathsf{BH}}(-k) = \hat{\delta}(k^2)\hat{\delta}(p \cdot k)(p \cdot \varepsilon)^2 \exp\left(-i\frac{k_{\mu}\varepsilon_{\nu}S^{\mu\nu}}{p \cdot \varepsilon}\right),$$

where $\qquad S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} p_{\rho} a_{\sigma}$

*Hat notation absorbs straightforward powers of 2π.

Kerr $\Leftarrow$ minimal coupling to gravity

Guevara, AO, Vines '18

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Compare to

$$\mathcal{M}_3^{(s,+)} = \frac{\mathcal{M}_3^{(0,+)}}{m^{2s}}[2|^{\odot 2s}\exp\left(-i\frac{k_\mu \varepsilon^+_\nu \bar{\sigma}^{\mu\nu}}{p_1\cdot\varepsilon^+}\right)|1]^{\odot 2s}$$

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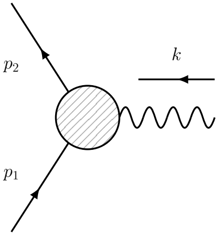

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Guevara, AO, Vines '18

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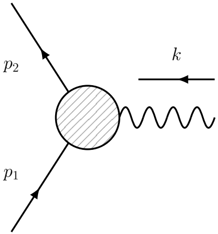

Matching spin-induced multipole structure!

complementary picture: 1-body EFT of Kerr by Levi, Steinhoff '15

match to Wilson coeffs by Chung, Huang, Kim, Lee '18

Spin exponentiation in covariant form

Covariant formulation:

Bautista, Guevara '19

$$\mathcal{M}_3^{(s)} = \mathcal{M}_3^{(0)} \varepsilon_2 \cdot \exp\left(-i \frac{k_\mu \varepsilon_\nu \Sigma^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \cdot \varepsilon_1$$

Lorentz generators:

$$(\Sigma^{\mu\nu})^{\sigma_1 \ldots \sigma_s}{}_{\tau_1 \ldots \tau_s} = \Sigma^{\mu\nu,\sigma_1}{}_{\tau_1} \delta^{\sigma_2}_{\tau_2} \ldots \delta^{\sigma_s}_{\tau_s} + \ldots + \delta^{\sigma_1}_{\tau_1} \ldots \delta^{\sigma_{s-1}}_{\tau_{s-1}} \Sigma^{\mu\nu,\sigma_s}{}_{\tau_s},$$

$$\Sigma^{\mu\nu,\sigma}{}_\tau = i[\eta^{\mu\sigma}\delta^\nu_\tau - \eta^{\nu\sigma}\delta^\mu_\tau]$$

Polarization tensors:

Guevara, AO, Vines '18, Chung, Huang, Kim, Lee '18

$$\varepsilon^{a_1 \ldots a_{2s}}_{p\mu_1 \ldots \mu_s} = \varepsilon^{(a_1 a_2}_{p\mu_1} \ldots \varepsilon^{a_{2s-1} a_{2s})}_{p\mu_s}, \qquad \varepsilon^{ab}_{p\mu} = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m}$$

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Spinor-helicity formulation: Guevara, AO, Vines '19

$$\mathcal{M}_3^{(s,+)} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} [2|^{\odot 2s} \exp\left(-i \frac{k_\mu \varepsilon^+_\nu \bar{\sigma}^{\mu\nu}}{p_1 \cdot \varepsilon^+}\right) |1]^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} [2|^{\odot 2s} \exp(-2k \cdot a) |1]^{\odot 2s}$$

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$$a^{\mu,}{}_\alpha{}^\beta = \frac{1}{2m^2} \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} \sigma_{\rho\sigma,\alpha}{}^\beta, \qquad a^{\mu,\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2m^2} \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} \bar{\sigma}_{\rho\sigma,}{}^{\dot{\alpha}}{}_{\dot{\beta}}$$

$$\sigma^{\mu\nu} = \tfrac{i}{2} \sigma^{[\mu} \bar{\sigma}^{\nu]}, \qquad \bar{\sigma}^{\mu\nu} = \tfrac{i}{2} \bar{\sigma}^{[\mu} \sigma^{\nu]}$$

(and tensor generalizations)

Spin quantization

Define Pauli-Lubanski vector operator $\Sigma_\lambda = \frac{1}{2m}\epsilon_{\lambda\mu\nu\rho}\Sigma^{\mu\nu}p^\rho$

Its 1-particle matrix elements are

$$\begin{aligned} S_{p\mu}^{\{a\}\{b\}} &= (-1)^s \varepsilon_p^{\{a\}} \cdot \Sigma_\mu \cdot \varepsilon_p^{\{b\}} \\ &= -\frac{s}{2m}\big\{\langle p^{(a_1}|\sigma_\mu|p^{(b_1}] + [p^{(a_1}|\bar{\sigma}_\mu|p^{(b_1}\rangle\big\}\epsilon^{a_2b_2}\ldots\epsilon^{a_{2s})b_{2s})} \end{aligned}$$

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Spin quantized explicitly:

$$\frac{\varepsilon_{p\{a\}} \cdot \Sigma^\mu \cdot \varepsilon_p^{\{a\}}}{\varepsilon_{p\{a\}} \cdot \varepsilon_p^{\{a\}}} = \begin{cases} s s_p^\mu, & a_1 = \ldots = a_{2s} = 1, \\ (s-1)s_p^\mu, & \sum_{j=1}^{2s} a_j = 2s+1, \\ (s-2)s_p^\mu, & \sum_{j=1}^{2s} a_j = 2s+2, \\ \ldots & \\ -s s_p^\mu, & a_1 = \ldots = a_{2s} = 2, \end{cases}$$

in terms of unit spin vector

$$\begin{aligned} s_p^\mu &= -\frac{1}{2m}\big\{\langle p_1|\sigma^\mu|p^1] + [p_1|\bar{\sigma}^\mu|p^1\rangle\big\} \\ &= \frac{1}{2m}\bar{u}_{p1}\gamma^\mu\gamma^5 u_p^1 = -\frac{1}{2m}\bar{u}_{p2}\gamma^\mu\gamma^5 u_p^2 \end{aligned}$$

$$p \cdot s_p = 0 \qquad s_p^2 = -1$$

Spin asymmetry of chiral reps

Puzzle:

two reps of $\mathcal{M}_3^{(s,+)} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} \langle 21 \rangle^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} [2|^{\odot 2s} e^{-2k \cdot a} |1]^{\odot 2s}$
seem to depend differently on a^μ

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Fix:

Guevara, AO, Vines '18

"divide" by

$$\lim_{s\to\infty} \varepsilon_2 \cdot \varepsilon_1 = \lim_{s\to\infty} \frac{1}{m^{2s}} \langle 2|^{\odot 2s} e^{k\cdot a} |1\rangle^{\odot 2s} = \lim_{s\to\infty} \frac{1}{m^{2s}} [2|^{\odot 2s} e^{-k\cdot a} |1]^{\odot 2s}$$

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Hint: Levi, Steinhoff '15

"spin-induced higher multipoles should naturally be considered in the body-fixed frame"

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Levi, Steinhoff '15

"spin-induced higher multipoles should naturally be considered in the body-fixed frame"

Solution:

Bautista, Guevara '19

Guevara, AO, Vines '19

also in Arkani-Hamed, Huang, O'Connell '19

must only compare states of same momentum!

Lorentz-boost exponentials

Bautista, Guevara '19
Guevara, AO, Vines '19
also in Arkani-Hamed, Huang, O'Connell '19

Consider $p_1 \to p_2$ boost:

$$p_2^\rho = \exp\big(\tfrac{i}{m^2} p_1^\mu k^\nu \Sigma_{\mu\nu}\big)^\rho{}_\sigma p_1^\sigma$$
$$|2^b\rangle = U_{12}{}^b{}_a \exp\big(\tfrac{i}{m^2} p_1^\mu k^\nu \sigma_{\mu\nu}\big)|1^a\rangle$$
$$|2^b] = U_{12}{}^b{}_a \exp\big(\tfrac{i}{m^2} p_1^\mu k^\nu \bar{\sigma}_{\mu\nu}\big)|1^a]$$

$$k^2 = (p_2 - p_1)^2 = 0$$
$$U_{12} \in \mathrm{SU}(2)$$

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$$k^2 = (p_2 - p_1)^2 = 0$$
$$U_{12} \in \mathrm{SU}(2)$$

Self-duality of $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ implies

$$\frac{i}{m^2} p_1^\mu k^\nu \sigma_{\mu\nu,\alpha}{}^\beta = k \cdot a_\alpha{}^\beta, \qquad \frac{i}{m^2} p_1^\mu k^\nu \bar\sigma_{\mu\nu,}{}^{\dot\alpha}{}_{\dot\beta} = -k \cdot a^{\dot\alpha}{}_{\dot\beta}$$

in terms of left- and right-handed reps of Pauli-Lubanski vector

$$a^{\mu,}{}_\alpha{}^\beta = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} \sigma_{\rho\sigma,\alpha}{}^\beta, \qquad a^{\mu,\dot\alpha}{}_{\dot\beta} = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} \bar\sigma_{\rho\sigma,}{}^{\dot\alpha}{}_{\dot\beta}$$

Spin exponentials from Lorentz boosts

Arbirary-spin reps boost as

$$|2\rangle^{\odot 2s} = e^{k\cdot a}\big\{U_{12}|1\rangle\big\}^{\odot 2s}, \qquad |2]^{\odot 2s} = e^{-k\cdot a}\big\{U_{12}|1]\big\}^{\odot 2s}$$

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---

* m^{2s} cancels due to $\langle p^a p^b\rangle = -[p^a p^b] = -m\epsilon^{ab}$.

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Back to spin dependence of 3-pt amplitude:*

$$\begin{aligned}\mathcal{M}_3^{(s,+)} &= \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\langle 21\rangle^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\{U_{12}\langle 1|\}^{\odot 2s} e^{-k\cdot a}|1\rangle^{\odot 2s} \\ &= \frac{\mathcal{M}_3^{(0)}}{m^{2s}}[2|^{\odot 2s} e^{-2k\cdot a}|1]^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\{U_{12}[1|\}^{\odot 2s} e^{-k\cdot a}|1]^{\odot 2s} \\ &\xrightarrow[s\to\infty]{} \mathcal{M}_3^{(0)} e^{-k\cdot a}\lim_{s\to\infty}(U_{12})^{\odot 2s} \quad \text{unambigiously!}\end{aligned}$$

a^μ is now classical (C-number) spin of Kerr BH

* m^{2s} cancels due to $\langle p^a p^b\rangle = -[p^a p^b] = -m\epsilon^{ab}$.

1PM with general spin dependence

Impulse formulae

Kosower, Maybee, O'Connell '18

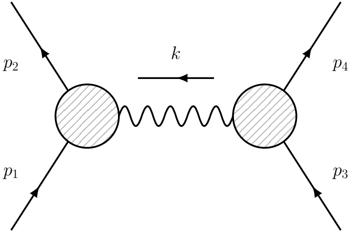
Maybee, O'Connell, Vines '19

LO impulses:

$$\Delta p_{\mathrm{a}}^{\mu} = \left\langle\!\!\left\langle \int \hat{d}^4 k\, \hat{\delta}(2p_{\mathrm{a}}\cdot k)\hat{\delta}(2p_{\mathrm{b}}\cdot k) k^{\mu} e^{-ik\cdot b/\hbar} i\mathcal{M}_4(k) \right\rangle\!\!\right\rangle$$

$$\Delta S_{\mathrm{a}}^{\mu} = \left\langle\!\!\left\langle \int \hat{d}^4 k\, \hat{\delta}(2p_{\mathrm{a}}\cdot k)\hat{\delta}(2p_{\mathrm{b}}\cdot k) e^{-ik\cdot b/\hbar} \times \left(-\frac{i}{m_{\mathrm{a}}^2} p_{\mathrm{a}}^{\mu} S_{\mathrm{a}}^{\nu} k_{\nu} \mathcal{M}_4(k) + \left[S_{\mathrm{a}}^{\mu}, i\mathcal{M}_4(k) \right] \right) \right\rangle\!\!\right\rangle$$

$$p_{\mathrm{a}} = \frac{1}{2}(p_1 + p_2)$$



$$p_{\mathrm{b}} = \frac{1}{2}(p_3 + p_4)$$

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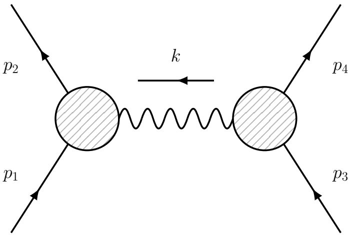


$$p_{\mathrm{b}} = \frac{1}{2}(p_3 + p_4)$$

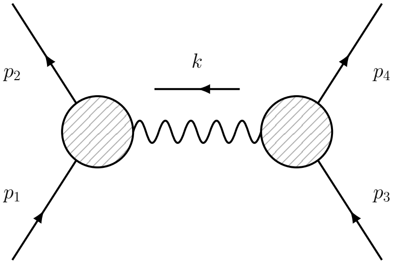
Net effect of $\langle\!\langle \ldots \rangle\!\rangle$:

$$k^{\mu} = \hbar \bar{k}^{\mu} \to 0, \qquad p_1^{\mu}, p_2^{\mu} \to m_{\mathrm{a}} u_{\mathrm{a}}^{\mu}, \qquad p_3^{\mu}, p_4^{\mu} \to m_{\mathrm{b}} u_{\mathrm{b}}^{\mu}$$

$$S_1^{\mu}, S_2^{\mu} \to m_{\mathrm{a}} a_{\mathrm{a}}^{\mu}, \qquad S_3^{\mu}, S_4^{\mu} \to m_{\mathrm{b}} a_{\mathrm{b}}^{\mu}$$

Holomorphic Classical Limit (HCL)

Cachazo, Guevara '17
Guevara '17



Idea: Replace $k^\mu = \hbar \bar{k}^\mu \to 0$ by non-zero on-shell $t = k^2 \to 0$

Indeed, $k^2 = 0 \quad \Rightarrow \quad p_i \cdot k = \mathcal{O}(t) = 0$

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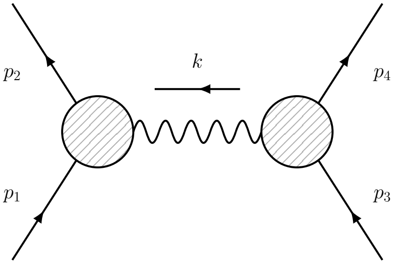

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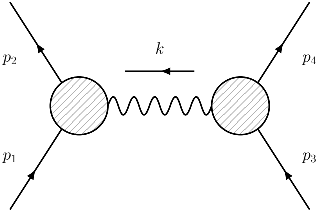
Indeed, $k^2 = 0 \quad \Rightarrow \quad p_i \cdot k = \mathcal{O}(t) = 0$

$$\mathcal{M}_4^{(s_\mathrm{a},s_\mathrm{b})}(p_1,-p_2,p_3,-p_4) = \frac{-1}{t}\sum_{\pm}\mathcal{M}_3^{(s_\mathrm{a})}(p_1,-p_2,k^\pm)\mathcal{M}_3^{(s_\mathrm{b})}(p_3,-p_4,-k^\mp) + \mathcal{O}(t^0)$$

4-pt "classical amplitude" from HCL

Guevara, AO, Vines '19



$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{p_{\rm a}\cdot p_{\rm b}}{m_{\rm a}m_{\rm b}} \to u_{\rm a}\cdot u_{\rm b}$$

$$\mathcal{M}_4 = \frac{-(\kappa/2)^2\gamma^2}{m_{\rm a}^{2s_{\rm a}-2}m_{\rm b}^{2s_{\rm b}-2}t}\Big((1-v)^2\big\{U_{12}\langle 1|\big\}^{\odot 2s_{\rm a}}e^{-k\cdot a_{\rm a}}|1\rangle^{\odot 2s_{\rm a}}\big\{U_{34}[3|\big\}^{\odot 2s_{\rm b}}e^{-k\cdot a_{\rm b}}|3]^{\odot 2s_{\rm b}}$$
$$+(1+v)^2\big\{U_{12}[1|\big\}^{\odot 2s_{\rm a}}e^{k\cdot a_{\rm a}}|1]^{\odot 2s_{\rm a}}\big\{U_{34}\langle 3|\big\}^{\odot 2s_{\rm b}}e^{k\cdot a_{\rm b}}|3\rangle^{\odot 2s_{\rm b}}\Big)$$

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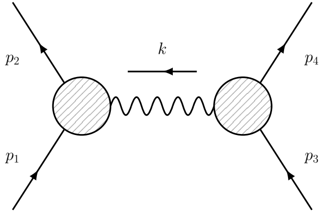

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Remove parity-oddness using

$$k\cdot a_{\rm a,b} = ik\cdot w * a_{\rm a,b}, \qquad [w * a_{\rm a,b}]_\mu = \frac{\epsilon_{\mu\nu\rho\sigma}a_{\rm a,b}^{\nu}p_{\rm a}^{\rho}p_{\rm b}^{\sigma}}{m_{\rm a}m_{\rm b}\gamma v}$$

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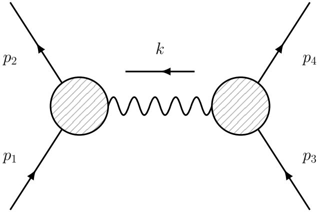

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$$\langle \mathcal{M}_4(k)\rangle = -\left(\frac{\kappa}{2}\right)^2 \frac{m_\mathrm{a}^2 m_\mathrm{b}^2}{k^2}\gamma^2 \sum_\pm (1\pm v)^2 \exp[\pm i(k\cdot w * a_0)], \quad a_0^\mu = a_\mathrm{a}^\mu + a_\mathrm{b}^\mu$$

4-pt scattering function

Guevara, AO, Vines '19

from momentum transfer/mismatch k^μ

$$\langle \mathcal{M}_4(k) \rangle = -\left(\frac{\kappa}{2}\right)^2 \frac{m_{\rm a}^2 m_{\rm b}^2}{k^2} \gamma^2 \sum_{\pm} (1 \pm v)^2 \exp[\pm i(k \cdot w * a_0)]$$

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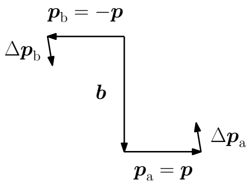
to impact parameter b^μ

$$\begin{aligned} \langle \mathcal{M}_4(b) \rangle &= \int \hat{d}^4 k \, \hat{\delta}(2p_\mathrm{a} \cdot k) \hat{\delta}(2p_\mathrm{b} \cdot k) e^{-ik\cdot b} \langle \mathcal{M}_4(k) \rangle \\ &= -G m_\mathrm{a} m_\mathrm{b} \frac{\gamma}{v} \sum_\pm (1 \pm v)^2 \log \sqrt{-(b \mp w * a_0)^2} \end{aligned}$$



$$b^2 < 0$$
$$b \cdot p_\mathrm{a} = 0$$
$$b \cdot p_\mathrm{b} = 0$$

eikonal Fourier transform e.g. in Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18

Linear and angular impulses from scattering function

Guevara, AO, Vines '19

$$\langle \mathcal{M}_4(b) \rangle = -Gm_{\rm a}m_{\rm b}\frac{\gamma}{v}\sum_{\pm}(1 \pm v)^2 \log\sqrt{-(b \mp w * a_0)^2}$$

$$\Delta p_{\rm a}^\mu = \left\langle\!\!\left\langle \int \hat{d}^4 k\, \hat{\delta}(2p_{\rm a}\cdot k)\hat{\delta}(2p_{\rm b}\cdot k) k^\mu e^{-ik\cdot b/\hbar} i\mathcal{M}_4(k) \right\rangle\!\!\right\rangle = -\frac{\partial}{\partial b_\mu}\langle \mathcal{M}_4(b)\rangle$$

---

*Relied on little-group $\mathrm{so}(3)$ algebra of $S_{\rm a}^\mu$ in rest frame of $p_{\rm a}$, i.e.

$$[S_{\rm a}^\mu, S_{\rm a}^\nu] = \frac{i}{m_{\rm a}}\epsilon^{\mu\nu\rho\sigma}p_{{\rm a}\rho}S_{{\rm a}\sigma} \quad \Rightarrow \quad [S_{\rm a}^\mu, \mathcal{M}_4] = \frac{i}{m_{\rm a}}\epsilon^{\mu\nu\rho\sigma}p_{{\rm a}\nu}S_{{\rm a}\rho}\frac{\partial \mathcal{M}_4}{\partial S_{\rm a}^\sigma}.$$

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$$\left.\left. \times \left(-\frac{i}{m_{\rm a}^2} p_{\rm a}^\mu S_{\rm a}^\nu k_\nu \mathcal{M}_4(k) + \left[S_{\rm a}^\mu, i\mathcal{M}_4(k) \right] \right) \right\rangle\!\!\right\rangle$$
$$\overset{*}{=} \frac{1}{m_{\rm a}^2} \left[p_{\rm a}^\mu a_{\rm a}^\nu \frac{\partial}{\partial b^\nu} - \epsilon^{\mu\nu\rho\sigma} p_{{\rm a}\nu} a_{{\rm a}\rho} \frac{\partial}{\partial a_{\rm a}^\sigma} \right] \langle \mathcal{M}_4(b) \rangle$$

---

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Guevara, AO, Vines '19

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$$\begin{aligned} \Delta a_\mathrm{a}^\mu &= \frac{1}{m_\mathrm{a}} \left\langle\!\!\left\langle \int \hat{d}^4 k \, \hat{\delta}(2p_\mathrm{a} \cdot k) \hat{\delta}(2p_\mathrm{b} \cdot k) e^{-ik\cdot b/\hbar} \right.\right. \\ &\qquad \left.\left. \times \left(-\frac{i}{m_\mathrm{a}^2} p_\mathrm{a}^\mu S_\mathrm{a}^\nu k_\nu \mathcal{M}_4(k) + \left[S_\mathrm{a}^\mu, i\mathcal{M}_4(k) \right] \right) \right\rangle\!\!\right\rangle \\ &\overset{*}{=} \frac{1}{m_\mathrm{a}^2} \left[p_\mathrm{a}^\mu a_\mathrm{a}^\nu \frac{\partial}{\partial b^\nu} - \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} a_{\mathrm{a}\rho} \frac{\partial}{\partial a_\mathrm{a}^\sigma} \right] \langle \mathcal{M}_4(b) \rangle \end{aligned}$$

Complete match to 1PM classical solution! Vines '17

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1PM classical solution for general spin orientations

Vines '17

Linear and angular impulses

$$\Delta p_{\mathrm{a}}^{\mu} = G m_{\mathrm{a}} m_{\mathrm{b}} \Re Z^{\mu}$$
$$\Delta a_{\mathrm{a}}^{\mu} = -\frac{G m_{\mathrm{b}}}{m_{\mathrm{a}}} \left[p_{\mathrm{a}}^{\mu} (a_{\mathrm{a}} \cdot \Re Z) + \epsilon^{\mu\nu\rho\sigma} (\Im Z_{\nu}) p_{\mathrm{a}\rho} a_{\mathrm{a}\sigma} \right]$$

in terms of an auxiliary complex vector

$$Z^{\mu} = \frac{\gamma}{v} \sum_{\pm} \big(1 \pm v\big)^2 [\eta^{\mu\nu} \mp i(*w)^{\mu\nu}] \frac{(b \mp w * a_0)_{\nu}}{(b \mp w * a_0)^2}$$

Z^{μ} automatic from scattering function $\langle \mathcal{M}_4(b) \rangle$

$$\frac{\partial}{\partial b^{\mu}} \langle \mathcal{M}_4(b) \rangle = -G m_{\mathrm{a}} m_{\mathrm{b}} \Re Z_{\mu}, \qquad \frac{\partial}{\partial a^{\mu}} \langle \mathcal{M}_4(b) \rangle = G m_{\mathrm{a}} m_{\mathrm{b}} \Im Z_{\mu}$$

Aligned-spin results at 2PM

Classical contributions from loops

- **Th:** classical from 2-massive-p. irreducible graphs with 1 massive prop. per loop   Neill, Rothstein '13

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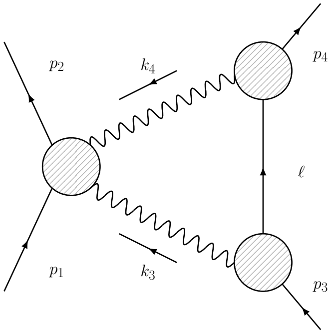

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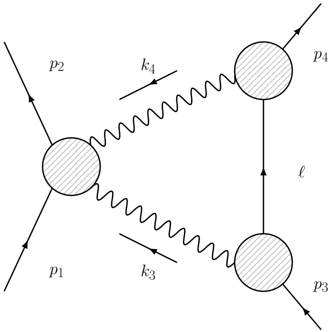

- 1 loop: boxes contribute to $\Delta p^{\mu}_{\mathrm{a,b}}$ — Kosower, Maybee, O'Connell '18
  but not to scattering angle θ — Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18

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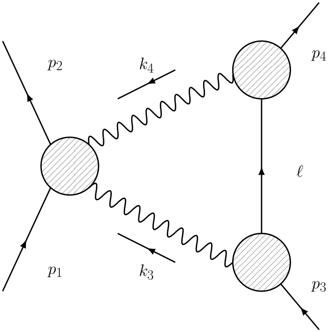

- 1 loop: boxes contribute to $\Delta p^{\mu}_{\mathrm{a,b}}$ — Kosower, Maybee, O'Connell '18
  but not to scattering angle θ
  Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18
- 2 loops: topologies with more massive props. contribute
  Bern, Cheung, Roiban, Shen, Solon, Zeng '19

2PM aligned-spin scattering angle from 1 loop

Guevara, AO, Vines '18

- Incoming spins $\perp$ to scattering plane
  $\Rightarrow$ outgoing spins stay aligned, $\Delta a_{\mathrm{a,b}} = 0$, scattering within plane
  $\Rightarrow$ scattering angle θ implies $\Delta p_{\mathrm{a,b}}$

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- Use known non-spinning formula from eikonal

$$2\sin\frac{\theta}{2} = \frac{-E}{(2m_a m_b \gamma v)^2}\frac{\partial}{\partial b}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{b}} \lim_{s_a,s_b\to\infty}\langle \mathcal{M}_4^{(s_a,s_b)}\rangle + \mathcal{O}(G^3)$$

Kabat, Ortiz '92; Akhoury, Saotome '13
Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18

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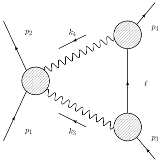

- Compute triangle coeffs in HCL

Cachazo, Guevara '17; Guevara '17

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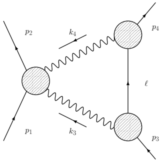

- Compute triangle coeffs in HCL

Cachazo, Guevara '17; Guevara '17

- Extract angular-momentum dependence from spin exponentials

2PM aligned-spin scattering angle result

Guevara, AO, Vines '18

$$\theta_{\triangleleft} = \pi G^2 E \frac{m_b}{2v^4} \frac{\partial}{\partial b} \oint_{R>1/v} \frac{dz}{2\pi i} \frac{(1-vz)^4}{(z^2-1)^{3/2}} \left| b - z a_{\mathrm{b}} - \frac{z-v}{1-vz} a_{\mathrm{a}} \right|^{-1}$$

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$$\theta^{\text{1-loop}} = \theta_{\triangleleft} + \theta_{\triangleright} = -\pi G^2 E \, \frac{\partial}{\partial b} \Big[m_{\mathrm{b}} f(a_{\mathrm{a}}, a_{\mathrm{b}}) + m_a f(a_{\mathrm{b}}, a_{\mathrm{a}}) \Big],$$

where

$$E = \sqrt{m_{\mathrm{a}}^2 + m_{\mathrm{b}}^2 + 2 m_{\mathrm{a}} m_{\mathrm{b}} \sqrt{1-v^2}},$$

$$f(\sigma, a) = \frac{1}{2a^2} \left[-b + \frac{(\jmath + \varkappa - 2a)^5}{4v\varkappa \big[(\jmath + \varkappa)^2 - (2va)^2 \big]^{3/2}} \right] + \mathcal{O}(\sigma^5),$$

$$\jmath = vb + \sigma + a, \qquad \varkappa = \sqrt{\jmath^2 - 4va(b + v\sigma)}$$

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$$\jmath = vb + \sigma + a, \qquad \varkappa = \sqrt{\jmath^2 - 4va(b + v\sigma)}$$

true at least through $\mathcal{O}(a^2)$, possibly wrong beyond $\mathcal{O}(a^4)$

Bini, Damour '18
Vines, Steinhoff, Buonanno '18

2PM aligned-spin scattering angle vs PN theory

$$\theta = \frac{GE}{v^2}\sum_{\pm}\frac{(1\pm v)^2}{b\pm(a_\mathrm{a}+a_\mathrm{b})} - \pi G^2 E\,\frac{\partial}{\partial b}\Big[m_\mathrm{b} f(a_\mathrm{a},a_\mathrm{b}) + m_a f(a_\mathrm{b},a_\mathrm{a})\Big] + \mathcal{O}(G^3)$$

(taken through $\mathcal{O}(G^2S^4)$)

- agrees with all PN results through NNLO S^2

Porto, Rothstein '06, '08; Levi '08, '10; Perrodin '10; Porto '10

Levi '11; Levi, Steinhoff '14 '15 '16

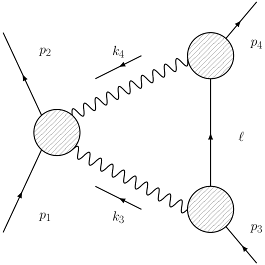
- conjectures new results at 4.5PN (NLO S^3) and 5PN (NLO S^4)

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
	N	1PN		2PN		3PN		4PN		5PN			
			LO SO		NLO SO		NNLO SO		NNNLO SO				
				LO S^2		NLO S^2		NNLO S^2		NNNLO S^2			
							LO S^3		NLO S^3				
								LO S^4		NLO S^4			
											LO S^5		NLO S^5
												LO S^6	

need up to
1PM / tree
2PM / 1-loop
3PM / 2-loop
4PM / 3-loop
5PM / 4-loop
6PM / 5-loop

table by Vines

Gravitational Compton amplitude



$$\begin{aligned}\mathcal{M}_4^{(s)}(p_1,-p_2,k_3^+,k_4^-) &= -\left(\frac{\kappa}{2}\right)^2 \langle 2|^{\odot 2s} \exp\left(-i\frac{k_{4\mu}\varepsilon_{4\nu}^- \sigma^{\mu\nu}}{p_1\cdot\varepsilon_4^-}\right)|1\rangle^{\odot 2s} \\ &= -\left(\frac{\kappa}{2}\right)^2 [2|^{\odot 2s} \exp\left(-i\frac{k_{3\mu}\varepsilon_{3\nu}^+ \bar{\sigma}^{\mu\nu}}{p_1\cdot\varepsilon_3^+}\right)|1]^{\odot 2s} \\ &= \left(\frac{\kappa}{2}\right)^2 \frac{\langle 4|1|3]^{4-2s}\big([13]\langle 42\rangle + \langle 14\rangle[32]\big)^{\odot 2s}}{(2p_1\cdot k_4)(2p_2\cdot k_4)(2k_3\cdot k_4)}\end{aligned}$$

- first appeared in — Arkani-Hamed, Huang, Huang '17
- problematic at $s \geq 4$ — double-copy aspects in Johansson, AO '19
- alternatives proposed in — Chung, Huang, Kim, Lee '18

Summary & outlook

- Spin exponentiation pattern inherent to Kerr BH

  Guevara, AO, Vines '18
  Bautista, Guevara '19
  Guevara, AO, Vines '19
  Arkani-Hamed, Huang, O'Connell '19

- 1PM match for general spin (to all orders in spin) Vines '17
- 2PM results for aligned spins:
  - match at presently known orders in spins

    Bini, Damour '18
    Vines, Steinhoff, Buonanno '18
  - conjectured results for higher orders in spins

    consistent with Siemonsen, Vines '19
- Used HCL,
  need better connections between classical-limit approaches

  Guevara '17
  Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18
  Cheung, Rothstein, Solon '18
  Kosower, Maybee, O'Connell '18
  Koemans Collado, Di Vecchia, Russo '19
  Bjerrum-Bohr, Cristofoli, Damgaard, Vanhove '19
  Maybee, O'Connell, Vines '19
  Damgaard, Haddad, Helset '19
  Kälin, Porto '19
- Open questions at higher orders in G and spin

MORE NEW RESULTS TO COME!

Thank you!

Backup slides

Spin supplementary condition

Covariant spin exponentiation:

$$\mathcal{M}_3^{(s)} = \mathcal{M}_3^{(0)} \varepsilon_2 \cdot \exp\left(-i \frac{k_\mu \varepsilon_\nu \Sigma^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \cdot \varepsilon_1$$

$$\varepsilon_2 = \exp\left(\frac{i}{m^2} p_1^\mu k^\nu \Sigma_{\mu\nu}\right) \tilde{\varepsilon}_1, \qquad \tilde{\varepsilon}_1 = U_{12}^{(s)} \varepsilon_1$$ — LG transform

$$\mathcal{M}_3^{(s)} = \mathcal{M}_3^{(0)} \tilde{\varepsilon}_1 \exp\left(-\frac{i}{m^2} p_1^\mu k^\nu \Sigma_{\mu\nu}\right) \exp\left(-i \frac{k_\mu \varepsilon_\nu \Sigma^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \varepsilon_1$$

$$= \mathcal{M}_3^{(0)} \tilde{\varepsilon}_1 \exp\left(-i \frac{k_\mu \varepsilon_\nu \Sigma_\perp^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \varepsilon_1$$

$$\Sigma_\perp^{\mu\nu} = \Sigma^{\mu\nu} + \frac{2}{m^2} p_1^{[\mu} \Sigma^{\nu]\rho} p_{1\rho} \qquad \Rightarrow \qquad p_{1\mu} \Sigma_\perp^{\mu\nu} = 0$$ — SSC

Angular momentum diff. operator

Witten '03

used in Cachazo, Strominger '14

Massless momentum k^μ

$$J^{\mu\nu} = \left[\lambda_k^\alpha \sigma^{\mu\nu,}{}_\alpha{}^\beta \frac{\partial}{\partial \lambda_k^\beta} + \tilde{\lambda}_{k\dot{\alpha}} \bar{\sigma}^{\mu\nu,\dot{\alpha}}{}_{\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_{k\dot{\beta}}}\right]$$

$$J_{\alpha\dot{\alpha},\beta\dot{\beta}} = 2i\left[\lambda_{k(\alpha} \frac{\partial}{\partial \lambda_k^{\beta)}} \epsilon_{\dot{\alpha}\dot{\beta}} + \epsilon_{\alpha\beta} \tilde{\lambda}_{k(\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}^{k\dot{\beta})}}\right] = \sigma^\mu_{\alpha\dot{\alpha}} \sigma^\nu_{\beta\dot{\beta}} J_{\mu\nu}$$

Massive extension

Conde, Joung, Mkrtchyan '16

$$J_{\alpha\dot{\alpha},\beta\dot{\beta}} = 2i\left[\lambda_{p(\alpha}^{\ a} \frac{\partial}{\partial \lambda_p^{\beta)a}} \epsilon_{\dot{\alpha}\dot{\beta}} + \epsilon_{\alpha\beta} \tilde{\lambda}_{p(\dot{\alpha}}^{\ a} \frac{\partial}{\partial \tilde{\lambda}_p^{\dot{\beta})a}}\right]$$

Consistency:

$\forall$ spin-0 function $f(p)$ $\qquad J_{\mu\nu} f(p) = L_{\mu\nu} f(p), \qquad L_{\mu\nu} = 2ip_{[\mu} \frac{\partial}{\partial p^{\nu]}}$

$$J_{\mu\nu} p_\rho = p_\sigma \Sigma^{\mu\nu,\sigma}{}_\rho, \qquad J_{\mu\nu} \varepsilon_\rho = \varepsilon_\sigma \Sigma^{\mu\nu,\sigma}{}_\rho, \qquad \Sigma^{\mu\nu,\rho}{}_\sigma = i[\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma]$$

Angular momentum diff. action

Guevara, AO, Vines '18

Differentiation action:

$$k_\mu \varepsilon_\nu^+ J^{\mu\nu} |p^a\rangle = k_\mu \varepsilon_\nu^- J^{\mu\nu} |p^a] = 0$$

$$\left(\frac{k_\mu \varepsilon_\nu^- J^{\mu\nu}}{p \cdot \varepsilon^-} \right)^j |p^a\rangle^{\odot 2s} = \frac{(2s)!}{(2s-j)!} |p^a\rangle^{\odot(2s-j)} \odot \left(\frac{|k\rangle\langle k p^a\rangle}{m x_-} \right)^{\odot j}$$

$$\left(\frac{k_\mu \varepsilon_\nu^+ J^{\mu\nu}}{p \cdot \varepsilon^+} \right)^j |p^a]^{\odot 2s} = \frac{(2s)!}{(2s-j)!} |p^a]^{\odot(2s-j)} \odot \left(\frac{|k][k p^a]}{m x_+} \right)^{\odot j}$$

Algebraic realization on $|p\rangle^{\odot 2s}$ and $|p]^{\odot 2s}$:

$$\left(i \frac{k_\mu \varepsilon_\nu^- J^{\mu\nu}}{p \cdot \varepsilon^-} \right)^{\odot j} = \begin{cases} \frac{(2s)!}{(2s-j)!} \left(\frac{|k\rangle\langle k|}{m x_-} \right)^{\otimes j} \odot \mathbb{I}^{\otimes 2s-j} & , \quad j \leq 2s \\ 0 & , \quad j > 2s \end{cases}$$

$$\left(i \frac{k_\mu \varepsilon_\nu^+ J^{\mu\nu}}{p \cdot \varepsilon^+} \right)^{\odot j} = \begin{cases} \frac{(2s)!}{(2s-j)!} \left(\frac{|k][k|}{m x_+} \right)^{\otimes j} \odot \mathbb{I}^{\otimes 2s-j} & , \quad j \leq 2s \\ 0 & , \quad j > 2s \end{cases}$$

connects to intuitive "spin-operator" terminology in Guevara '17

3-pt amplitudes as exponentials

Guevara, AO, Vines '18

$$\langle 1^a 2^b \rangle = [1^a 2^b] + \frac{1}{m x_+}[1^a k][k\, 2^b], \qquad m x_+ = \sqrt{2}(p_1 \cdot \varepsilon^+)$$

$$\Updownarrow$$

$$[1^a 2^b] = \langle 1^a 2^b \rangle - \frac{1}{m x_-}\langle 1^a k \rangle \langle k\, 2^b \rangle, \qquad m x_- = \sqrt{2}(p_1 \cdot \varepsilon^-)$$

$$[1^a 2^b]^{\odot 2s} = \left[\langle 1^a 2^b \rangle - \frac{\langle 1^a k \rangle \langle k\, 2^b \rangle}{m x_-} \right]^{\odot 2s} = \langle 1^a |^{\odot 2s} \left[\sum_{j=0}^{2s} \binom{2s}{j} \left(- \frac{|k\rangle \langle k|}{m x_-} \right)^j \right] |2^b \rangle^{\odot 2s}$$

$$= \langle 1^a |^{\odot 2s} \exp\left(i \frac{k_\mu \varepsilon_\nu^- J_2^{\mu\nu}}{p_2 \cdot \varepsilon^-} \right) |2^b \rangle^{\odot 2s} = \langle 2^b |^{\odot 2s} \exp\left(i \frac{k_\mu \varepsilon_\nu^- J_1^{\mu\nu}}{p_1 \cdot \varepsilon^-} \right) |1^a \rangle^{\odot 2s}$$

$$\mathcal{M}_3^{(s)}(\overline{1}, \underline{2}, k^-) = \frac{x_-^2}{m^{2s-2}} [12]^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} \langle 2|^{2s} \exp\left(i \frac{k_\mu \varepsilon_\nu^- J^{\mu\nu}}{p \cdot \varepsilon^-} \right) |1\rangle^{2s}$$

$$\mathcal{M}_3^{(s)}(\overline{1}, \underline{2}, k^+) = \frac{x_+^2}{m^{2s-2}} \langle 12 \rangle^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} [2|^{2s} \exp\left(i \frac{k_\mu \varepsilon_\nu^+ J^{\mu\nu}}{p \cdot \varepsilon^+} \right) |1]^{2s}$$

Angular momentum in soft theorem and Kerr BH

Soft theorem:* Cachazo, Strominger '14

$$\mathcal{M}_{n+1} = \sum_{i=1}^{n} \left[\frac{(p_i \cdot \varepsilon)^2}{p_i \cdot k} - i \frac{(p_i \cdot \varepsilon)(k_\mu \varepsilon_\nu J_i^{\mu\nu})}{p_i \cdot k} - \frac{1}{2} \frac{(k_\mu \varepsilon_\nu J_i^{\mu\nu})^2}{p_i \cdot k} \right] \mathcal{M}_n + \mathcal{O}(k^2)$$

$$= \sum_{i=1}^{n} \frac{(p_i \cdot \varepsilon)^2}{p_i \cdot k} \left[1 - i \frac{k_\mu \varepsilon_\nu J_i^{\mu\nu}}{p_i \cdot \varepsilon} - \frac{1}{2} \left(\frac{k_\mu \varepsilon_\nu J_i^{\mu\nu}}{p_i \cdot \varepsilon} \right)^2 \right] \mathcal{M}_n + \mathcal{O}(k^2)$$

Energy tensor of Kerr BH: Vines '17

$$T^{\mu\nu}(k) = \delta(p \cdot k) p^{(\mu} \exp(-ia * k)^{\nu)}{}_\rho p^\rho + \mathcal{O}(G) \qquad \Rightarrow$$

$$\varepsilon_k^{\mu\nu} T_{\mu\nu}(k) = \delta(k^2) \delta(p \cdot k)(p \cdot \varepsilon)^2 \left[1 - i \frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p \cdot \varepsilon} - \frac{1}{2} \left(\frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p \cdot \varepsilon} \right)^2 + \mathcal{O}(k^3) \right],$$

where $\quad p^\mu = m u^\mu, \qquad S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} p_\rho a_\sigma$

*Omitting prefactors of $-i(\kappa/2)^{n-2}$ in $\mathcal{M}_n$, where $\kappa = \sqrt{32\pi G}$.