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Relativistic Transport Equations

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Cosmology, Particle Physics and Phenomenology - CP3

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Today

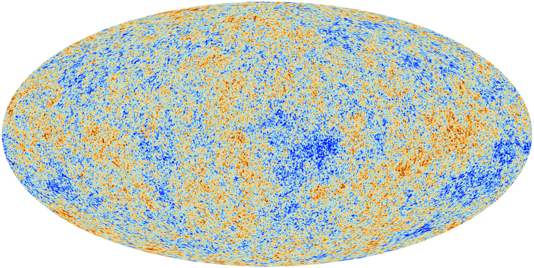
Outline

Introduction

Computing the relativistic evolution of a given distribution of particles is a common problem in cosmology

- Analysis of comic structure
  - ➔ Physics of the cosmic microwave background
  - ➔ Formation of the large scale structure
  - ➔ . . .
- In cosmology all information (transported via particles) must first reach our detectors
  - ➔ Cosmic microwave background observations
  - ➔ Galaxy observations
  - ➔ Transport of supernovae neutrinos
  - ➔ . . .

Introduction



- Collisions with electrons at reionisation: Quantum Mechanics
- Transport along non-trivial geodesics (for example gravitational lensing): General Relativity

Relativity plus Quantum Field Theory

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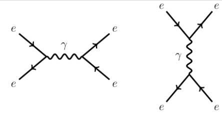
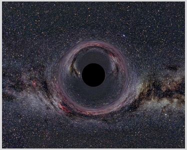
Microscopic Scale



Mesoscopic Scale

Macroscopic Scale



- Quantum description in the local Minkowski space on microscopic scale
- Semi-classical particle description on mesoscopic scale
- Relativistic description of classical particles on macroscopic scale

Theory is well established for photons (cosmic microwave background)

Fermion Transport

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- Number operator: $N_{rs}(p, p') \equiv a_r^\dagger(p) a_s(p')$ where $a_s(p')$ are creation/annihilation operators of helicity states
- Distribution function: $\langle\Psi|N_{rs}(p, p')|\Psi\rangle \equiv \underline{\delta}(p - p') f_{rs}(p)$
- We identify f_{rs} with a semi-classical particle distribution on the mesoscopic scale

Quantum Transport Equations

$$\underline{\delta}(0)\frac{\mathrm{d}}{\mathrm{d}t} f_{rs} = \langle\Psi|\frac{\mathrm{d}N_{rs}}{\mathrm{d}t}|\Psi\rangle = i\langle\Psi|[H, N_{rs}]|\Psi\rangle \equiv \underline{\delta}(0)\, C[f_{rs}(t, p)]$$

Computation of $C[f_{ss'}(t, p)]$ involves typical scattering matrix elements weighted by the distribution functions

Observer Dependance

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In contrast to massless photons, the helicity (and f_{rs}) is observer dependant

- Define the observer independent $F_{\mathfrak{a}}{}^{\mathfrak{b}}(p) \equiv \sum_{rs} f_{rs}(p) u_{s,\mathfrak{a}}(p) \bar{u}_r^{\mathfrak{b}}(p)$ with $u_{s,\mathfrak{a}}(p)$ the helicity spinors
- We define the parameters
  - ➔ Intensity $I \equiv f_{++} + f_{--}$
  - ➔ Circular polarisation $V \equiv f_{++} - f_{--}$
  - ➔ Linear polarisation $Q_\pm \equiv \sqrt{2} f_{\pm\mp}$
  - ➔ Polarisation vector $Q^\mu \equiv Q_+ \epsilon_+^\mu + Q_- \epsilon_-^\mu$
  - ➔ The combined vector $\mathcal{Q}^\mu \equiv Q^\mu + VS^\mu$
- F can be decomposed as $F = \frac{I}{2}\left(M + \not{p}\right) + \frac{M}{2}\gamma^5\gamma_\mu \mathcal{Q}^\mu + p^\mu \tilde{\Sigma}_{\mu\nu}\mathcal{Q}^\nu$ with $\widetilde{\Sigma}^{\mu\nu} \equiv -\frac{\gamma^5}{4}[\gamma^\mu, \gamma^\nu]$
- I and $\mathcal{Q}^\mu$ are individually observer dependant, while both helicity and linear polarisation are not.

The Macroscopic Evolution

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- Both I and $\mathcal{Q}^\mu$ are transported trivially in local Minkowski space in the absence of collisions
- In general relativity particles are parallel transported along the non-trivial geodesics
  - ➔ Obtain a relativistic counterpart by contracting the local vector $\mathcal{Q}^\mu$ with the tetrad
  - ➔ Compute the impact of the relativistic dynamics on the distribution functions
  - ➔ The collision term is added locally, taking into account the proper time of the particles

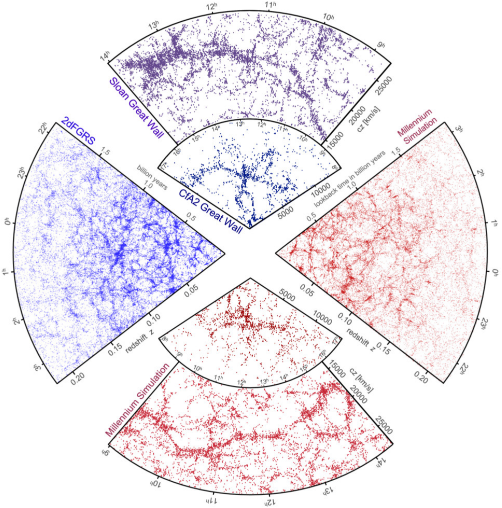
Relativistic Boltzmann Equation

$$\frac{\partial \mathcal{Q}^\alpha(\tau, \boldsymbol{x}, \boldsymbol{p})}{\partial \tau} + \frac{P^i}{P^0} \frac{\partial \mathcal{Q}^\alpha(\tau, \boldsymbol{x}, \boldsymbol{p})}{\partial x^i} + \frac{\mathrm{d}p^i}{\mathrm{d}\tau} \frac{\partial \mathcal{Q}^\alpha(\tau, \boldsymbol{x}, \boldsymbol{p})}{\partial p^i} + \Gamma^\alpha_{\gamma\beta} \frac{P^\gamma}{P^0} \mathcal{Q}^\beta(\tau, \boldsymbol{x}, \boldsymbol{p}) = \frac{E}{P^0} C^\alpha_{\mathcal{Q}}$$

Conclusions

- We have derived a relativistic transport formalism for polarised fermions
- Polarisation of the fermion gas is described by a vector $\mathcal{Q}^\alpha$ (instead of a polarisation tensor $F^{\alpha\beta}$ for bosons)
- Polarisation can exist in thermal equilibrium
- Polarisation is naturally induced by chiral interactions and appears as circular or linear polarisation dependant on the observer
- Charged interactions tend to wash out polarisation

The Large Scale Structure



Relativistic N-Body Simulations

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- The large scale structure of the Universe is expected to be one of the major cosmological probes in the near future
- The precision of large-scale structure data is increasing and will soon surpass the predictive power of the CMB data
- However, current simulations are usually performed using Newtonian gravity
- First relativistic simulations: gevolution, COSIRA

Gauge Freedom of General Relativity

- The gauge defines the coordinates
- The gauge specifies the dynamical equations

What Does a Simulation Compute?

1. Compute the density: $\rho_{\mathsf{N}} = \frac{1}{a^3} \sum_{\text{particles}} m \, \delta_{\mathrm{D}}^{(3)}(\boldsymbol{x} - \boldsymbol{x}_p)$
2. Compute the Newtonian potential: $\boldsymbol{\nabla}^2 \Phi_{\mathsf{N}} = -4\pi G \bar{a}^2 \rho_{\mathsf{N}}$
3. Move the particles: $\left(\frac{\partial}{\partial \eta} + \frac{\dot{a}}{a} \right) \boldsymbol{v}_{\mathsf{N}} = \boldsymbol{\nabla} \Phi_{\mathsf{N}}$

The Metric Including Linear Perturbations

$$\mathrm{d}s^2 = a^2 \Big(-(1+2A)\mathrm{d}\eta^2 - 2\partial^i B \, \mathrm{d}x_i \mathrm{d}\eta + \left[\delta^{ij}(1+2H_{\mathrm{L}}) + 2D^{ij} H_{\mathrm{T}} \right] \mathrm{d}x_i \mathrm{d}x_j \Big)$$

The full density $\rho = (1 - 3H_{\mathrm{L}})\rho_{\mathsf{N}}$ takes deformation of space into account

The N-Body Gauge ($B = v$, $H_{\mathrm{L}} = 0$)

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- Set $H_{\mathrm{L}} = 0$: No volume perturbations
  - ➔ Simulation does compute the relativistic density
- Set $B = v$: $\nabla^2 \Phi = -4\pi G \bar{a}^2 \bar{\rho} \delta$
  - ➔ The Bardeen potential is computed according to GR in the N-body simulation

Relativistic Corrections

$$\left(\frac{\partial}{\partial \eta} + \frac{\dot{a}}{a} \right) \boldsymbol{v} = \boldsymbol{\nabla} \Phi + \boldsymbol{\nabla} \gamma$$

$$\gamma = \ddot{H}_{\mathrm{T}} + \frac{\dot{a}}{a} \dot{H}_{\mathrm{T}} - 8\pi G a^2 p \Pi$$

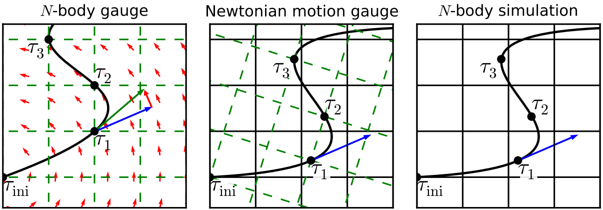
with the total anisotropic stress Π
In ΛCDM γ vanishes as $H_{\mathrm{T}} = 3\zeta$ is the comoving curvature perturbation and therefore conserved

Newtonian N-Body Simulations in N-Body Gauge

- Motion of particles is correctly simulated in a Newtonian N-body simulation
- Particle positions must be interpreted on the N-body gauge metric
- Radiation and late time anisotropic stresses are ignored

General Relativity

- Gauge theory with gauge dependant forces
  - ➔ Trajectories depend on the gauge



Using the Newtonian Motion Gauges

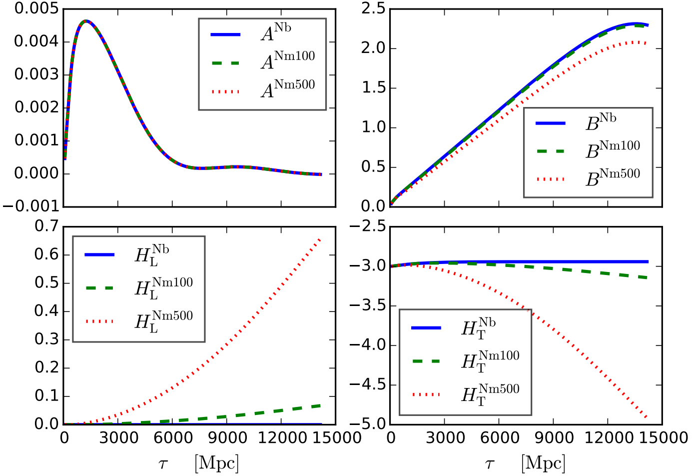
The Newtonian motion gauges decouple the full relativistic evolution

- Into the non-linear but Newtonian collapse of matter
  - ➔ Can be simulated by existing N-body codes
- And the relativistic but linear analysis of the underlying space-time
  - ➔ Can be implemented in existing Boltzmann codes

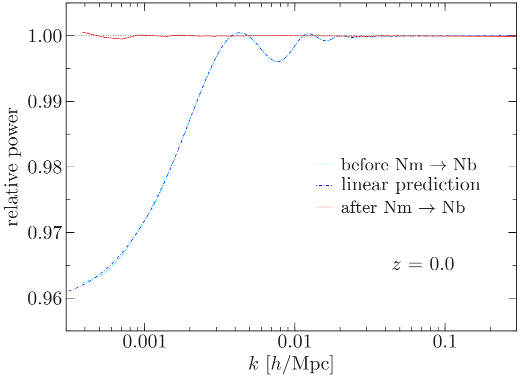
Recipe for a Relativistic Simulation

- Set up the relativistic Newtonian motion gauge initial conditions
- Run a Newtonian particle simulation to compute the dark matter evolution
- Solve for the Newtonian motion gauge space-time in a Boltzmann code (e.g. in CLASS)
- Interpret the trajectories on this space-time

The Newtonian Motion Gauge Metric

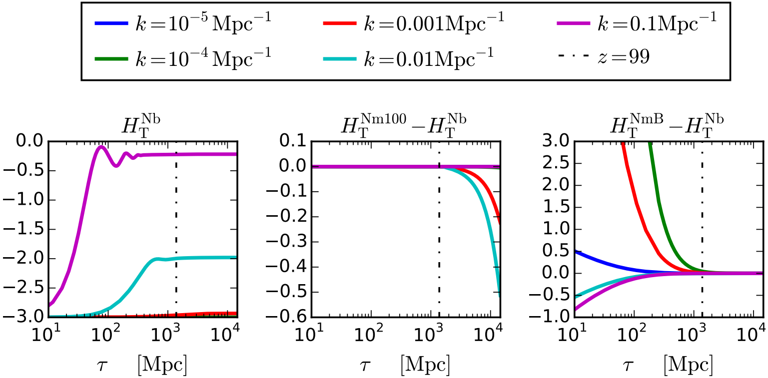


A Simulation in Newtonian Motion Gauge



Both numerical advantages and disadvantages compared to hybrid simulations (COSIRA, gevolution)

Backscaling in the Newtonian Motion Gauges



Conclusions

- Newtonian motion gauges allow a consistent embedding of Newtonian simulations in general relativity
- Numerically efficient and simple to use
- ΛCDM simulations using back-scaled initial conditions are in agreement with linear relativity if interpreted on the N-body gauge space-time
- Potential to include baryons, massive neutrinos, modified gravity, ...

`Thank You For Your Attention`

Example: Decaying Dark Matter