"No-hair" and uniqueness results for analogue black holes

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Outline

Introduction

Hawking radiation

[Hawking 75, Unruh 76]

- semiclassical approximation for quantum fields in a black hole space-time $\rightarrow$ vacuum for an infalling observer $\neq$ vacuum for an observer at infinity
  $\rightarrow$ production of pairs of particles
- thermal radiation at $T = T_H = \kappa/(2\pi)$

Is HR observable?

- solar-mass black hole $\Rightarrow T_H \approx 6 \times 10^{-8}$K
- cosmic microwave background temperature ≈ 2.7K

  $\Rrightarrow$ unlikely to be observable

Analogue Gravity

Main idea *[Unruh 81]*

- sound waves in a non-relativistic fluid at rest
  $\Rightarrow$ massless excitations $\partial_t^2 \phi - \Delta\phi = 0$

- non-uniform flow
  $\rightarrow \Box\phi = 0$ in a curved space-time metric (*analogue metric*)
  $\rightarrow$ analogue Killing horizon $\Leftrightarrow$ transonic flow

Why analogue gravity?

- observe the analogue version of the Hawking effect

- explore possible phenomenologies from quantum gravity effects (e.g., dispersion) *[Jacobson 91, Unruh 95]*

- understand condensed matter phenomena through the analogy *[Finazzi et al. 15]*

Non-linear effects

- nonlinear/backreaction terms different from those of QFT+GR $\rightarrow$ the correspondence breaks down
- however,
  - possible similarities $\rightarrow$ how far can the analogy extend?
  - interesting phenomena from a condensed matter/cold atoms point of view
- most studied example: "black-hole laser effect" *[Corley and Jacobson 99, Leonhardt and Philbin 08]*
  experimental realisation: *[Steinhauer 14]*

Black-hole laser

- linear order: dynamical instability → exponential growth of unstable modes
- non-linear behavior: variety of possible outcomes *[Michel and Parentani 13, Michel and Parentani 15, de Nova et al. 15]*
  - saturation on a stable (non-lasing) stationary solution
  - periodic emission of superposed soliton trains
  - seemingly aperiodic emission of solitons
- behavior of black-hole and white-hole flows separately currently unclear

Uniqueness results for black-hole flows

BH flows in BEC: setup

model and assumptions

- (1+1)D Bose-Einstein (quasi-)condensate (BEC)
- dilute and weakly interacting
- correlation length $\gg$ other length scales
- repulsive two-body interactions

$\Rightarrow$ Gross-Pitaevskii equation (GPE):

$$i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi + V(x)\psi + a^2(x)\left|\psi\right|^2 \psi$$

$V(x)$: (stationary) external potential

$a^2(x) > 0$: two-body interactions

BH flows in BEC: setup

Toy-model for analytical calculations

- $V(x) = \theta(-x)V_- + \theta(x)V_+$
- $a(x) = \theta(-x)a_- + \theta(x)a_+$
- $\left(a_+^2 - a_-^2\right)\left(V_+ - V_-\right) < 0$

$\Rightarrow$ solutions with a homogeneous density

$$\rho_0 = \frac{V_+ - V_-}{a_-^2 - a_+^2}$$

Stationary equation

general stationary configuration:

$$\psi(x,t) = e^{-i\omega t}\sqrt{\rho(x)}e^{i\int^x u(y)dy}$$

ω: angular frequency
ρ: atomic density
u: local velocity

conservation of the number of atoms $\Rightarrow \rho(x)u(x) = J$

resulting equation on ρ:

$$\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}} = 2\left(V - \omega\right) + 2a^2\rho + \frac{J^2}{\rho^2}$$

Solutions in homogeneous backgrounds (1)

Homogeneous a and $V \Rightarrow$ integration over x

$$\left(\partial_x \rho\right)^2 = 4a^2 \left(\rho(x) - \rho_1\right)\left(\rho(x) - \rho_2\right)\left(\rho(x) - \rho_3\right),$$

$(\rho_1, \rho_2, \rho_3) \in \mathbb{C}^3$ such that

$$\begin{cases} \rho_1 + \rho_2 + \rho_3 = 2\left(\omega - V\right)/a^2 \\ \rho_1 \rho_2 \rho_3 = J^2/a^2 \end{cases}$$

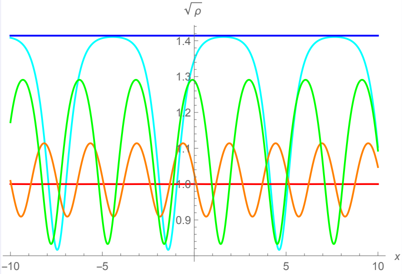
bounded solutions require

$$J^2 \leq J_{\max}^2 = \frac{8\left(\omega - V\right)^2}{27a^4}$$

and $(\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3$

Solutions in homogeneous backgrounds (2)

Periodic solutions



- $\rho_1 \leq \rho_2 \leq \rho_3 \to$ solution oscillates between ρ_1 and ρ_2
- $\rho_2 \approx \rho_1 \Rightarrow$ small-amplitude oscillations over a supersonic flow
- $\rho_2 \approx \rho_3 \Rightarrow$ soliton train over a subsonic flow

Characterization of BH flows

Conditions on BH flows

- continuity of ρ and ρ'
- asymptotically homogeneous
- transonic flow
- Mach number increases along the direction of the flow

6 degrees of freedom $-$ (1+2) asymptotic conditions $-$ 2 matching condition $\Rightarrow$ 1-parameter families of solutions, relation $\omega(J)$

In fact, only one solution at fixed J

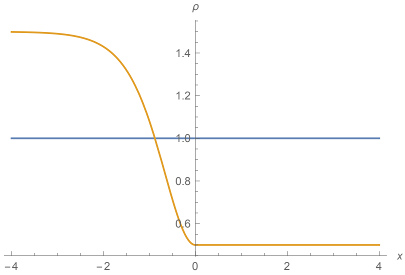
BH flows: two types of solutions

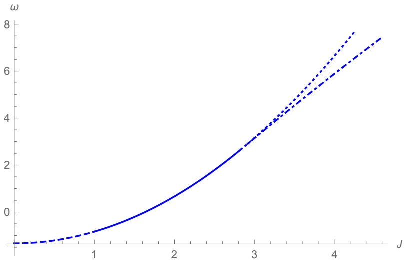
Homogeneous solution

- $\rho_{3,-} = \rho_{2,-} = \rho_{2,+} = \rho_{1,+}$
- $\rho(x) = \rho_0$
- transonic iff $a_- \rho_0^{3/2} > |J| > a_+ \rho_0^{3/2}$

Waterfall solution

- $\rho_{3,-} = \rho_{2,-}$, $\rho_{2,+} = \rho_{1,+} = \rho_{1,-}$,
- half-soliton matched with a homogeneous solution
- requires $|J| > a_- \rho_0^{3/2}$



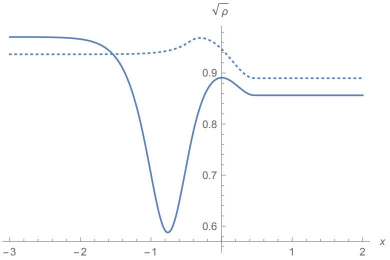


Smooth potentials: numerical results

Potentials of the form

$$f(x) = \begin{cases} f_- & x < -\sigma_f/2 \\ \frac{f_+ + f_-}{2} + \frac{f_+ - f_-}{\sigma_f} x & -\frac{\sigma_f}{2} \leq x \leq \frac{\sigma_f}{2} \\ f_+ & x > \sigma_f/2 \end{cases},$$

$f \in \{a^2, V\}$
"hairy" BH flows found for $\sigma_{a^2} > \sigma_V$, but seem to be unstable



Stability analysis

The Bogoliubov-deGennes equation

Perturbations in $\psi(x,t) = \psi_0(x)\left(1+\phi(x,t)\right)$
$\rightarrow$ BdG equation:

$$i\partial_t \phi = -\frac{1}{2}\partial_x^2 \phi - \frac{\partial_x \psi_0}{\psi_0}\partial_x \phi + a^2(x)\left|\psi_0^2\right|\left(\phi + \phi^*\right)$$

homogeneous background solution $\Rightarrow$

$$i\left(\partial_t + v_0 \partial_x\right)\phi = -\frac{1}{2}\partial_x^2 \phi + a^2(x)\rho_0\left(\phi + \phi^*\right)$$

basis of solutions of the form $\phi(x,t) = U e^{-i\omega t + ikx} + V^* e^{i\omega t - ikx}$
$\rightarrow$ dispersion relation:

$$\left(\omega - v_0 k\right)^2 = a^2 \rho_0 k^2 + \frac{k^4}{4}$$

Decay of linear perturbations: Idea of the calculation (1)

- find a basis of *in* modes $e^{-i\omega t}\phi_\omega^{(j)}(x)$
- expand the initial perturbation on this basis:

$$\phi(x,0) = \int \sum_j a_\omega^{(j)} \phi_\omega^{(j)}(x) d\omega$$

- evolve the solution in time:

$$\phi(x,t) = \int e^{-i\omega t} \sum_j a_\omega^{(j)} \phi_\omega^{(j)}(x) d\omega$$

- expand each *in* mode into plane waves:

$$\phi(x,t) = \int e^{-i\omega t} \sum_j a_\omega^{(j)} \left(\sum_l b_\omega^{(j,l)} e^{ik_\omega^l x} \right) d\omega$$

Decay of linear perturbations: Idea of the calculation (2)

- choice of the space of "acceptable" initial perturbations: no divergence in $a_\omega^{(i)}$ nor its derivatives
- late times: integral dominated by two contributions
  → points where $\left|b_\omega^{(j,l)}\right| \to \infty$
  → saddle points where $x \frac{dk_\omega^{(l)}}{d\omega} = t$
- singularities in $b_\omega^{(j,l)}$: only for $\omega \to 0$; come in pairs which cancel each other → no contribution to leading order
- saddle points at $\omega = \pm\omega_{\max}$ → Gaussian integration → decay in $O(t^{-3/2})$
- (possibly logarithmic factors from the Gaussian integration)

Whitham's modulation theory: Main ideas

[A. Kamchatnov, Nonlinear Periodic Waves and Their Modulations: An Introductory Course, 2000]

- goal: find approximate solutions of a (1+1)d PDE which either
  - vary slowly with space and time
  - show "fast", quasi-periodic oscillations modulated over a "slow" scale
- separation of scales → *averaging procedure*
- average the *conservation laws* over the "fast" scale → equations of evolution for locally conserved quantities
- the homogeneous GPE is integrable in the inverse scattering sense → infinite number of conserved quantities

Integrability in the AKNS sense

Integrability in the AKNS sense $\Leftrightarrow$ compatibility condition of the linear system

$$\begin{cases} \partial_x \psi_\lambda(x,t) = U(x,t;\lambda)\psi_\lambda(x,t) \\ \partial_t \psi_\lambda(x,t) = V(x,t;\lambda)\psi_\lambda(x,t) \end{cases}$$

(+technical conditions)

- $\psi_\lambda(x,t)$: two-component vector
- $U(x,t;\lambda)$, $V(x,t;\lambda)$: 2 by 2 matrices
- λ: complex parameter ("spectral parameter")

Explicitly,

$$\forall \lambda \in \mathbb{C}, [U(x,t;\lambda), V(x,t;\lambda)] = \partial_x V(x,t;\lambda) - \partial_t U(x,t;\lambda)$$

AKNS and conserved quantities

consider two solutions ϕ and φ

define $g \equiv \phi_1 \varphi_1$ and $P \equiv -\left(\phi_1 \varphi_2 - \phi_2 \varphi_1\right)^2 / 4$
$\rightarrow P$ depends only on λ

Conservation law:

$$\forall \lambda, \partial_t \left(\frac{U_{11}}{g} \right) - \partial_x \left(\frac{V_{11}}{g} \right) = 0$$

expansion in powers of $\lambda \rightarrow$ infinite number of conserved quantities

From exact conservation to slow modulation (1)

- two ways to apply the above for modulated solutions:
  • do the full analysis exactly $\rightarrow$ difficult to extract precise results
  • use that the solution varies little on small scales $\rightarrow$ define an effective local g
- the "local" and "global" g are related by a factor $\sqrt{P}$ (defined locally)
- we consider solutions for which P is a polynomial $\rightarrow$ the (exact) conservation laws give evolution equations on its roots λ_i
  Riemann invariants

From exact conservation to slow modulation (2)

- Whitham evolution equations:

$$\partial_t \lambda_i + v_i\left(\{\lambda_j\}\right) \partial_x \lambda_i = 0$$

$$v_i\left(\{\lambda_j\}\right) = \frac{\oint \frac{V_{11}(x,t;\mu)}{\sqrt{P(x,t;\mu)}g(x,t;\lambda_i)} d\mu}{\oint \frac{U_{11}(x,t;\mu)}{\sqrt{P(x,t;\mu)}g(x,t;\lambda_i)} d\mu}$$

- $\Rrightarrow$ the problem reduces to finding characteristics
- suitable ansatz $\Rightarrow$ known relation between $\{\lambda_i\}$ and ψ

The AKNS scheme for the homogeneous GPE: main steps

- find U and V for the GPE

$$U = \begin{pmatrix} -i\lambda & ia\psi \\ -ia\psi^* & i\lambda \end{pmatrix}, V = \begin{pmatrix} -i\lambda^2 - i\frac{a^2}{2}\left|\psi^2\right| & ia\lambda\psi - \frac{a}{2}\partial_x\psi \\ -ia\lambda\psi^* & i\lambda^2 + i\frac{a^2}{2}\left|\psi^2\right| \end{pmatrix}$$

- ansatz for g and P:

$$g(x,t;\lambda) = ia\psi(x,t)\,(\lambda - \mu(x,t))$$

$$P(\lambda) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4$$

justification a posteriori by recovering the solutions we are looking for

The AKNS scheme for the homogeneous GPE: main steps

- find the relationships between λ_i and the physical solution

$$\rho_3 = \left(\lambda_4 + \lambda_3 - \lambda_1 - \lambda_2\right)^2 / (4a^2)$$

$+$ permutations

- characteristic velocities:

$$v_i = \frac{1}{2}\left(\frac{L}{\partial_{\lambda_i} L} - s_1\right)$$

- wavelength of the periodic solutions:

$$L = \frac{a^2}{\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}} K\left(\frac{(\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1)}{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\right)$$

Scale-invariant solutions

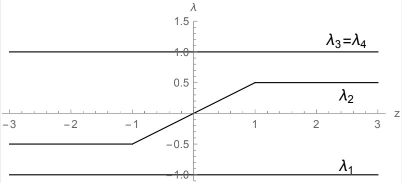
the initial configuration introduces no scale ⇒ scale-invariant solutions $\lambda_i(z \equiv x/t)$

Whitham's modulation equations:

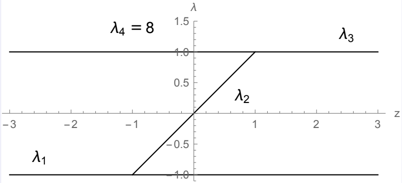
$$(v_i - z)\,\frac{d\lambda_i}{dz} = 0 \Rrightarrow v_i = z \text{ or } \frac{d\lambda_i}{dz} = 0$$

asymptotically homogeneous ⇒ two Riemann invariant are equal

simple wave (SW)

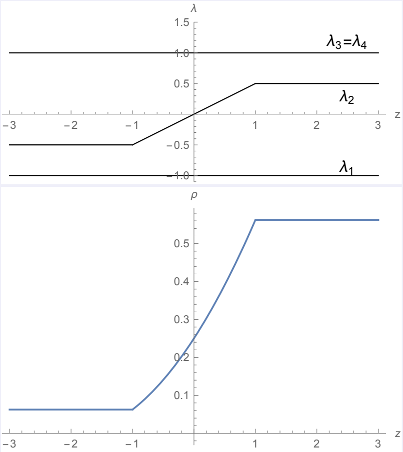


dispersive shock wave (DSW)



Scale-invariant solutions

simple wave (SW)



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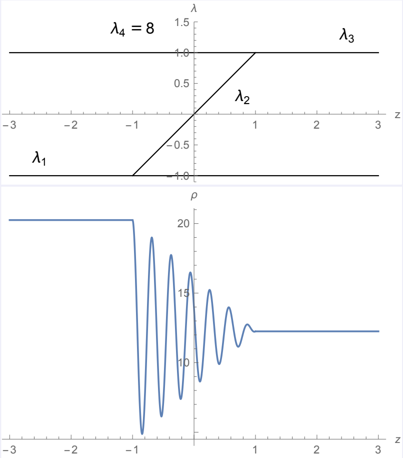
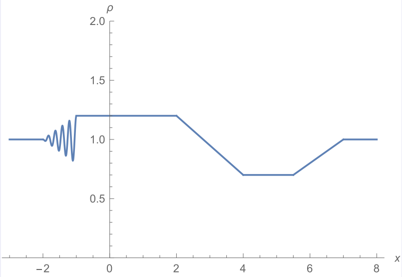


Solutions in step-like potentials

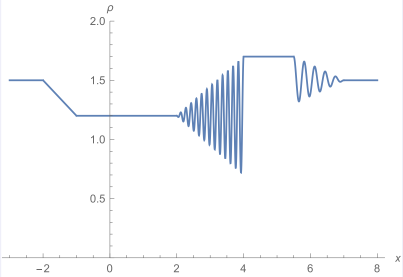
- initial conditions preserving scale invariance:
  - homogeneous density perturbation
  - piecewise-constant density perturbation with a single discontinuity at $x = 0$
- look for solutions with $\rho = \rho_0$ for $z \in I$, where I is an interval containing 0 $\rightarrow$ local convergence to a homogeneous solution.
- linear order: two outgoing solutions for $x > 0$; one for $x < 0$
  $\Rightarrow$ look for (NL) solutions with two waves downstream and one wave upstream
- each wave can be a DSW or a SW $\Rightarrow$ 8 possibilities

Homogeneous initial condition: two series of solutions

1 DSW and 2 SW

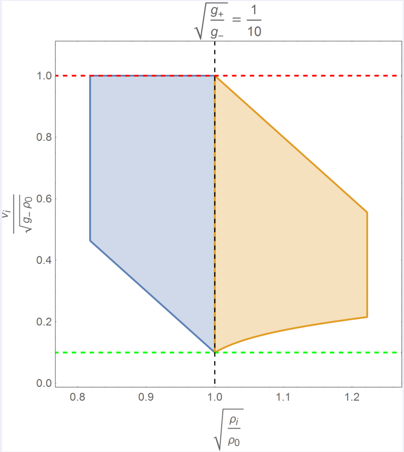


2 DSW and 1 SW



Homogeneous initial condition: analytical results

Domain of existence



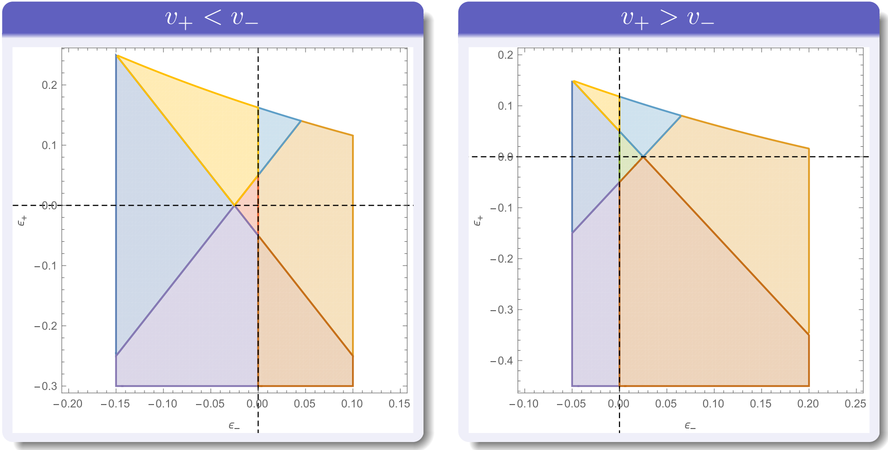
- dimensionless parameters: $\gamma \equiv a_-/a_+$, $\eta \equiv \sqrt{\rho_i/\rho_0}$, $\nu \equiv v_i/(\sqrt{\rho_0} a_-)$
- solution with 1 DSW and 2 SW $\Leftrightarrow$

$$\begin{cases} \frac{\gamma-1}{\gamma+1} < \eta < 1 \\ \frac{1}{\gamma} + 2(1-\eta) < \nu < 1 \end{cases}$$

- solution with 2 DSW and 1 SW $\Leftrightarrow$

$$\begin{cases} 1 < \eta < \frac{\gamma+1}{\gamma-1} \\ 2\frac{\eta}{\gamma} - \gamma^{-2}\left(\eta + \frac{\eta}{\gamma} - 1\right)^{-1} < \nu < 3 - 2\eta \end{cases}$$

Step-like initial configuration: Domains of existence



Conclusion: A solution always exists when starting sufficiently close to a homogeneous BH flow

Numerical results (1)

What can be learned from numerical resolution:

- validity of Whitham's theory
- changes due to smooth initial conditions and/or potentials
- time-evolution of configurations outside the domain of existence of solutions with 3 SW/DSW

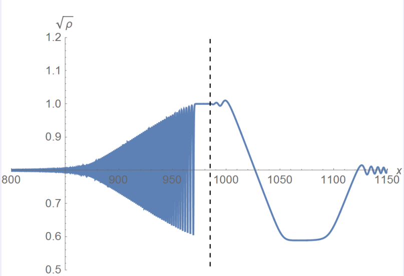
Numerical integration of the GPE with homogeneous or step-like initial conditions

$\rightarrow$ very good agreement with the solutions of Whitham's equations at late times (as expected)

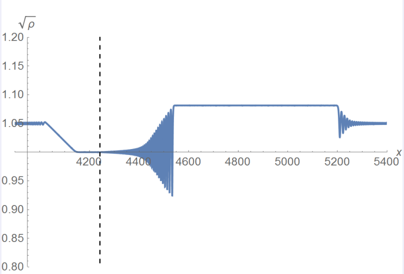
$\rightarrow$ earlier times: solution of Whitham's equations + perturbations in $O(t^{-\alpha})$, $\alpha \approx 3/2$

Numerical results (2)

1 DSW and 2 SW



2 DSW and 1 SW



Numerical results (3)

- smooth initial conditions and/or potentials $\rightarrow$ same late-time solutions
  (but formation time of the SW and DSW approximately linear in their scales of variation)

- Initial conditions outside the domain of existence of these solutions $\Rightarrow$ variety of behaviors
  $\rightarrow$ convergence to the homogeneous solution through the emission of overlapping non-linear waves
  $\rightarrow$ "hairy BH" with an extended, stationary soliton train
  $\rightarrow$ periodic emission of solitons

Conclusions

- BH flows of the GPE and KdV equation: described by a few conserved quantities (uniqueness)
- the homogeneous (or near-homogeneous) solutions are linearly stable
- non-linear stability: local convergence to the homogeneous (or near-homogeneous) solution through the emission of 3 non-linear waves + decaying perturbations
- white-hole flows: generically unstable $\rightarrow$ non-asymptotically uniform and/or nonstationary solutions

Outlook

- prove the non-linear stability of BH flows without relying on the hypotheses of the Whitham theory
- extend the analytical results to smooth potentials and more general perturbations
- analyze the stability of BH waterfall solutions
- understand the generation of soliton trains by WH flows

Thank you for your attention!