Stochastic perturbation of integrable systems

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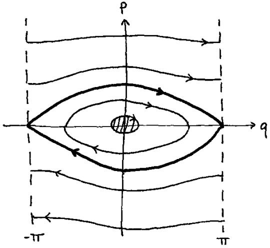
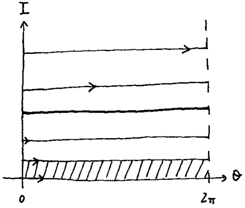
Integrable systems

N independent constants of motion





Action $(I_1, ...I_N)$ and Angle $(\theta_1, ..., \theta_N)$ variables

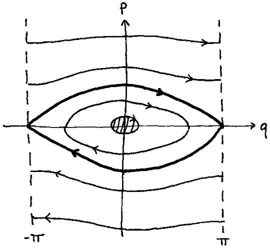
Action-angle representation

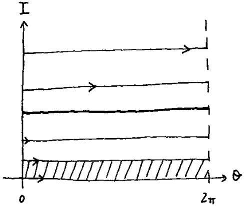
$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned}$$

$$\dot{I}_i = -\frac{\partial H}{\partial \theta_i}$$

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i(\mathbf{I})$$

Flow is *laminar*, restricted to tori $I_i = const$**,** $\theta_i = \omega_i t$



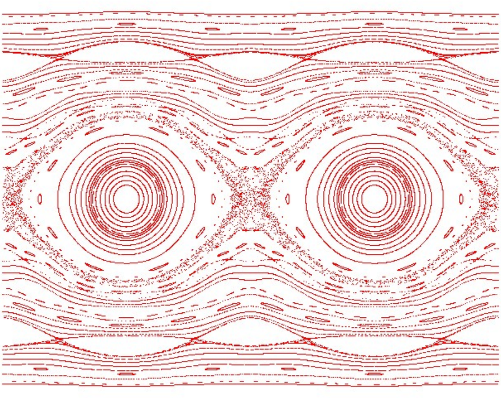


Topology: stationary points and separatrices

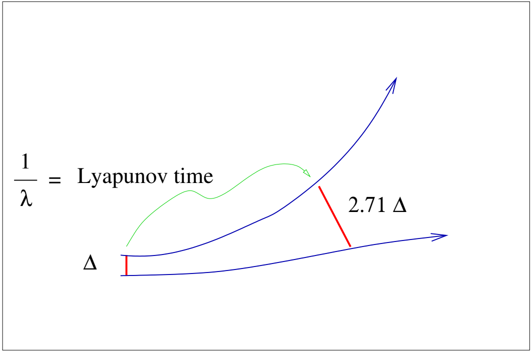
Integrable Kepler trajectories, perturbed by other planets

Perturbations: the KAM result

the example of a periodically kicked pendulum



Chaos. The Lyapunov instability



Lyapunov time (million Years) Moser

Mercury	1.4M
Venus	7.2 M
Earth	4.8M
Mars	4.5M
Jupiter	8.4 M
Saturn	6.4M
Uranus	7.5M
Neptune	6.7M

with some grains of salt...

The loss of integrability means that not much may be done analytically, and there is also trouble numerically

More importantly, it poses questions about stability

Our strategy:

We perturb with weak, additive noise

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + \varepsilon^{\frac{1}{2}} \xi_i(t)\end{aligned}$$

mostly consider the case in which the $\xi(t)$ are white noises:

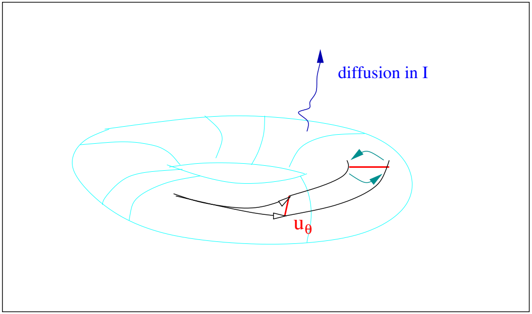
$$\langle \xi(t) \rangle = 0, \qquad \text{and} \qquad \langle \xi(t)\xi(t') \rangle = 2\delta(t - t').$$

In the action-angle variables, the noise is no longer additive, and reads:

$$\dot{I}_i = \varepsilon^{\frac{1}{2}} \sum_k \{I_i, q_k\} \xi_k(t)$$

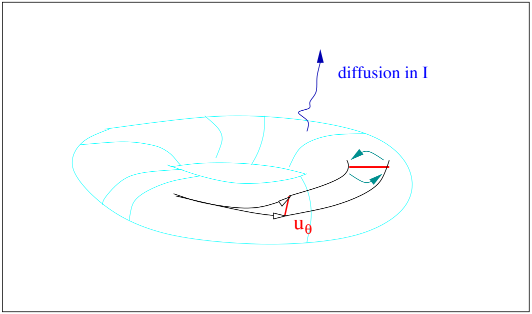
$$\dot{\theta}_i = \omega_i + \varepsilon^{\frac{1}{2}} \sum_k \{\theta_i, q_k\} \xi_k(t)$$

Surprise: a Lyapunov instability appears



two trajectories subjected to the *same* noise diverge exponentially

To leading order, everything happens



along the flow (on the tori)

Quick flash of calculation

$$
\begin{aligned}
\dot{u}_{\theta i} &= \sum_j \frac{\partial^2 H}{\partial I_j \partial I_i}\, u_{Ij} \\
\dot{u}_{Ii} &= \sum_{kj} \frac{\partial^2 q_k}{\partial \theta_i \partial \theta_j}(\varepsilon^{-\alpha} t)\, \xi_k(t)\, u_{\theta j}
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}_{\theta} &= \frac{d^2 H}{dI^2}\, u_I \\
\dot{u}_I &= \rho(t) u_\theta
\end{aligned}
$$

with $\overline{\rho(t)\rho(t')} = \delta(t-t')\Lambda_{II\theta\theta}$.

$$
\Lambda_{II\theta\theta} = \overline{\left(\frac{\partial^2 q}{\partial \theta^2}\right)^2} = \overline{\left(\frac{\ddot{q}}{\omega(I)^2}\right)^2}
$$

Quick flash of calculation

$$\dot{u}_\theta = \frac{d^2 H}{dI^2} u_I$$
$$\dot{u}_I = \rho(t) u_\theta$$

$$\ddot{u}_\theta = \frac{d^2 H}{dI^2} \rho(t) u_\theta$$

put $z = \frac{\dot{u}_\theta}{u_\theta}$

$$\ddot{u}_\theta - \frac{d^2 H}{dI^2} \rho(t)\, u_\theta \quad ; \quad \dot{z} - z^2 = \rho(t)$$

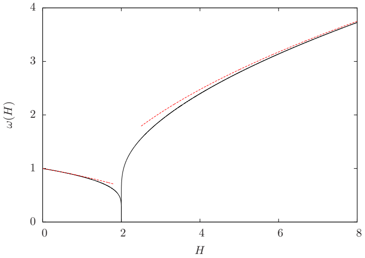
we connect with Halperin, Gardner-Derrida, Mallick-Marcq, ...

Result:

$$\lambda = \omega'(I)\langle z\rangle = \left(\frac{3}{2}\right)^{1/3} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{6})} \left(\varepsilon\overline{(\ddot{q})^2}\left(\frac{1}{\omega}\frac{d\omega}{dH}\right)^2\right)^{1/3} .$$

in terms of the average of $\ddot{q}$ over a cycle

Separatrices: the pendulum $H = \frac{1}{2}\, p^2 + 1 - \cos q$



$$\lambda = \omega'(I)\langle z\rangle = \left(\frac{3}{2}\right)^{1/3} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{6})} \left(\varepsilon\overline{(\ddot{q})^2}\left(\frac{1}{\omega}\frac{d\omega}{dH}\right)^2\right)^{1/3}.$$

Separatrices: the pendulum

$$H = \tfrac{1}{2}\, p^2 + 1 - \cos q$$

For $\delta \equiv |H - 2|$, one may compute

$$\omega(\delta \to 0) \simeq \frac{\pi}{|\log \delta|} \to 0$$

$$\frac{1}{\omega}\frac{d\omega}{dH} \sim \frac{1}{\delta |\log \delta|} \to \infty.$$

$$\varepsilon^{-\frac{1}{3}}\lambda \to \infty.$$

The angle α of the Lyapunov vector with the torus

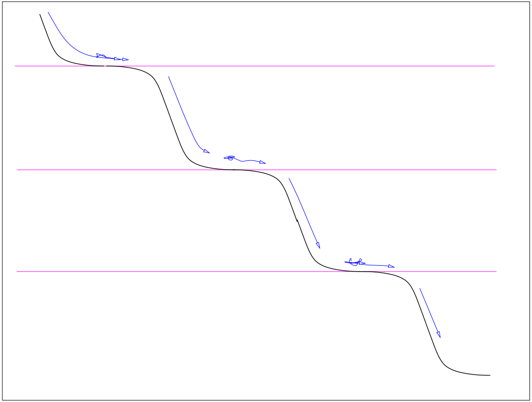
$$\alpha = \arctan z = \arctan\left(\frac{u_I}{u_\theta}\right)$$

The angle α follows a Langevin equation:

$$\dot{\alpha} \approx -\frac{dV}{d\alpha} + \frac{1}{\tau\omega'}\,\xi(t)$$

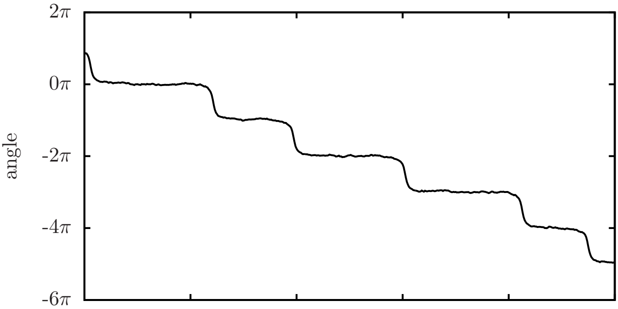
$$V(\alpha) = \frac{\omega'}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha - \ small\right)$$

Evolution of the angle

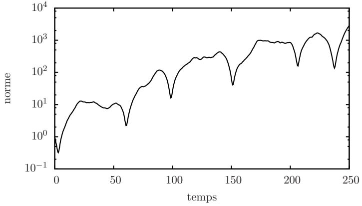


is punctuated by fast phase-slips

A numerical example



Meanwhile, the modulus grows steadily *between slips*



One finds a universal result: $\langle t_{slip} \rangle = 1.81\ \tau_\lambda$

A polymer in laminar flow

Chertkoff, Kolokov, Lebedev and Turitsyn

Polymer statistics in a random flow with mean shear

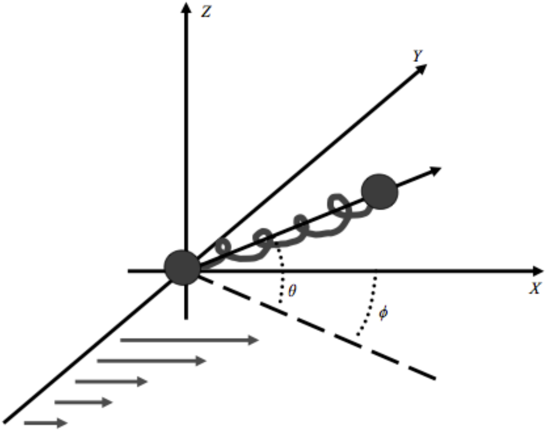


FIGURE 1. Polymer orientation geometry.

The problem is closely related to a classical problem in solid state

localization in a pseudorandom potential (band-edge)

$$\ddot{u}_{\theta i} + \sum_j u_{\theta j} = \ddot{u}_{\theta i} + \sum_j \hat{H}_{ij} u_{\theta j}$$

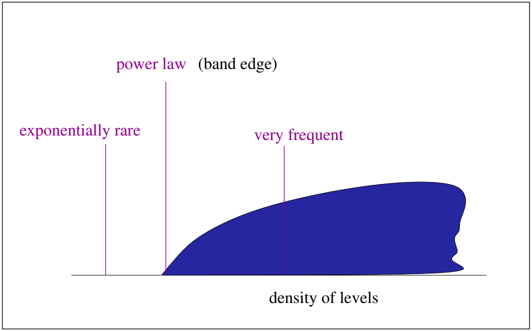
Shrödinger eigenvalue equation

$u_{\theta i} \to \psi_i$ **and** $t \to x$,

$$\nabla^2 \psi + \hat{\mathbf{H}} \psi = e\psi$$

density of zeroes $e < 0 \rightarrow$ number of phase-slips per unit time

Gardner-Derrida



... many things to learn from this vast literature

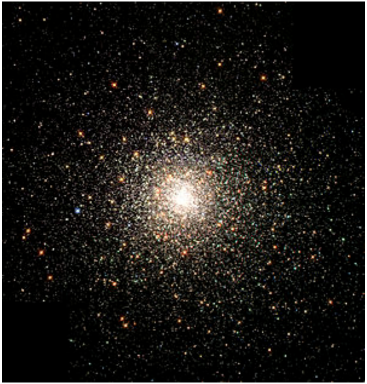
Weakly perturbed integrable models: mimicking complicated perturbations with stochastic ones

1. **An integrable mean-field**

2. **Perturbation in planetary systems**

A regime beyond KAM, and beyond the Nekhoroshev, for which there is no theory (?)

A spherical mean field, time-dependent granularity is the nonintegrable perturbation



Integrable Kepler trajectories, perturbed by other planets

Lyapunov time (million Years) Moser

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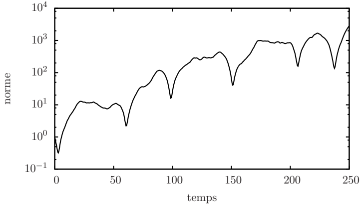
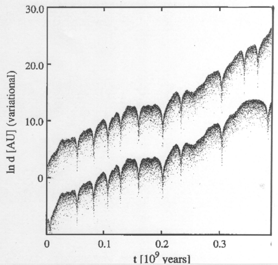
with some questions that I have...

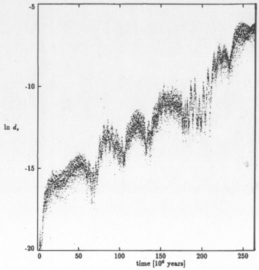
A simple system perturbed by noise



$$\langle t_{slip} \rangle = 1.81\ \tau_\lambda$$

Pluto and Saturn *phase-slips* **J. Wisdom**





Fermi-Pasta-Ulam chain

$$H = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + \frac{1}{2}(x_i - x_{i+1})^2 + V(x_i - x_{i+1})^4 \right)$$

Is way beyond the KAM regime, has a large Lyapunov exponent

and yet is famously slow in thermalizing!

Fermi-Pasta-Ulam chain (Benettin)

$$H = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + \frac{1}{2}(x_i - x_{i+1})^2 + V(x_i - x_{i+1})^4 \right)$$

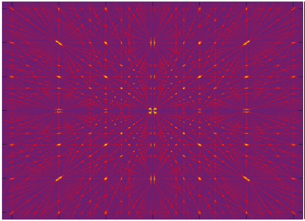
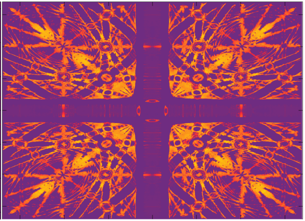
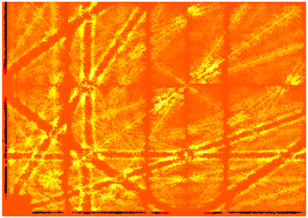
$$V(r) = \frac{r^2}{2} + \alpha \frac{r^3}{3} + \beta \frac{r^4}{4} = \frac{1}{4\alpha^2}(e^{2\alpha r} - 1 - 2\alpha r) + small$$

Integrable + small: a case we can understand

Fermi-Pasta-Ulam chain (Benettin)

***Just like the planets*, most of the Lyapunov instability is ‘on the torus’**

Froeschlé model

$$H = \sum_{i=1}^{N} \frac{I_i^2}{2} + I_0 + \frac{\epsilon(N+2)}{1 + \frac{1}{N+2}\sum_{i=0}^{N} \cos\theta_i} + \sum_i \cos\theta_i$$

Froeschlé model

$$\dot{\theta}_0 = 1,$$
$$\dot{\theta}_i = I_i$$
$$\dot{I}_i = -\epsilon \sin\theta_i - \epsilon \sin\theta_i \xi(t),$$

with the ‘effective noise’

$$\xi(t) = \left(1 + \frac{1}{N+2}\sum_i \cos\theta_i\right)^{-2} - 1.$$

each degree of freedom is a perturbed pendulum

Estimate the noise characteristics for random I_i with variance β

$$\xi(t) = -\frac{2}{N+2}\sum_{i=0}^{N}\cos\theta_i + \frac{3}{(N+2)^2}\left(\sum_{i=0}^{N}\cos\theta_i\right)^2 + \mathcal{O}(N^{-3}).$$

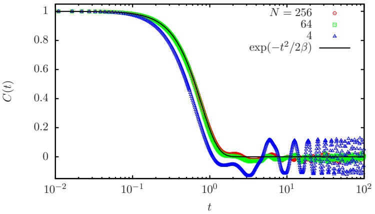
$$\langle\xi\rangle = \frac{3}{2}\frac{N+1}{(N+2)^2} \simeq \frac{3}{2N},$$

$$\langle\xi^2\rangle = 2\frac{N+1}{(N+2)^2} \simeq \frac{2}{N}$$

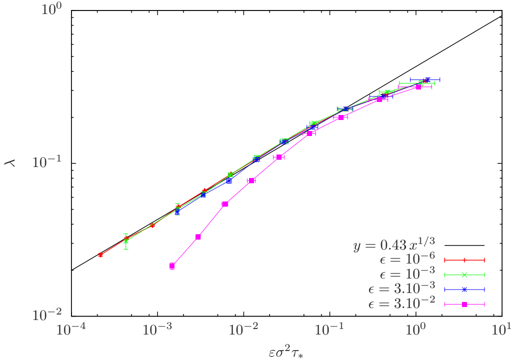
$$\sigma^2 = \langle\xi^2\rangle - \langle\xi\rangle^2 \simeq \frac{2}{N}$$

Estimate the noise characteristics

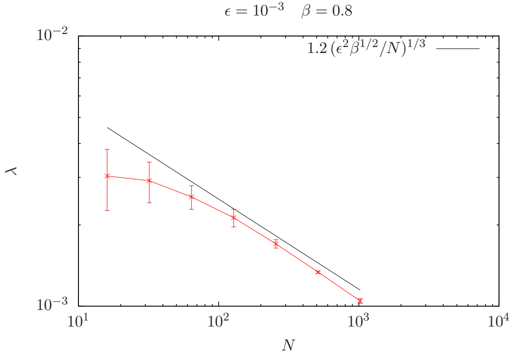
Autocorrelation



Using the true $\xi(t)$ as a noise on a separate system



we get a rather good agreement



$$4 < N < 8192$$

Diffusion of the eccentricity of Mercury, slightly different runs

Laskar

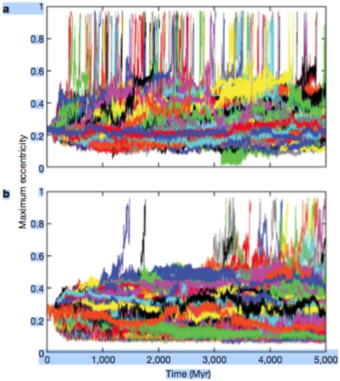


Figure 1 | Mercury's eccentricity over 5 Gyr. Evolution of the maximum

- **Suggests a statistical treatment might be illuminating also in this very different regime**

Ordinary diffusion, Taylor diffusion and Lyapunov regimes