The geodesic light-cone coordinates, an adapted system for (weak) lensing calculations

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Talk based on collaboration with:
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Outline of the Talk

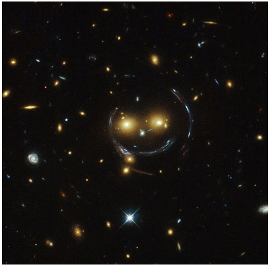
1. Motivations
2. Lens equation and GLAMER
3. The Jacobi Map
4. Geodesic light-cone coordinates
5. Illustration in a LTB model
6. Conclusions

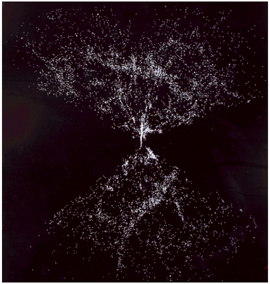
Talk mostly based on *Fanizza, Nugier 2014*, 1408.1604.

Motivations : lensing applications

Lensing probes any gravitating matter (but needs modelling) :

- estimation of galaxy mass (presence of arcs, flux ratios),
- probing dark matter substructure (time delays),
- galaxies distribution ('galaxy-galaxy' lensing).



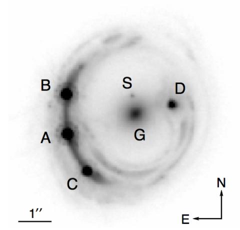


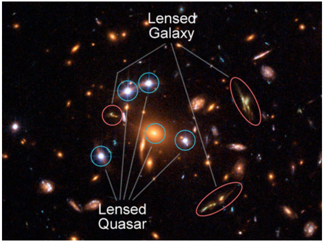
Statistical lensing informs on "lenses" distribution (once we know well our source distribution) :

- estimates the number density of compact objects in the dark halo of the Milky Way,
- gives the redshift evolution of galaxy number density,
- number density of clusters.

Lensing probes cosmology (with good modelling of lens/structure) :

- gives H_0 indep. from distance ladder and on cosmic scale (time delays in multiple images),
- constrains matter density param.s of large-scale structure,
- constrains the EoS of dark energy (when combined with CMB),
- bounds the bias parameter.





Lensing magnifies and hence shows faint old objects (as a telescope) :

- makes possible detection of galaxies at $z > 4$ (with cluster lenses),
- allows studying the composition of galaxy sources (in arcs),
- enables planet detection (with microlensed light curves),
- recently SN II lensed into a cross.

Schneider, Kochanek, Wambsganss "Grav. Lensing : Strong, Weak and Micro."

There are different approaches in lensing (not always obviously related) :

- the **lens equation** (e.g. used in ray shooting),
- the **Jacobi map** approach,
- the derivation from **Fermat principle**.

Questions :

- Can we get **easy lensing expressions** in the “GLC” coordinates ?
- Then can it help to compute them in **difficult cases** ? (e.g. LTB model)

Hypotheses

So the **notions to introduce** first are :

- the Jacobi map and lensing observables,
- "GLC" = "geodesic light-cone" coordinates.

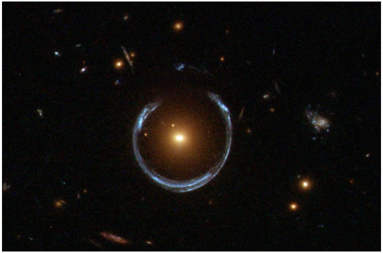
We assume the following :

H1 : **Jacobi map** formalism $\Rightarrow$ free from **thin lens** approximation, but depends on the **Born** (angles of deviation are small, typically $<$ arcmin) and **geometrical optics** approximations (wavelength irrelevant).

H2 : **no caustics** on the sky (as GLC coordinates break down), namely that the lenses under study are not 'strong' $\Rightarrow$ **WEAK LENSING !**

H2 is stronger than H1.

The Glenco Group



My activity in Bologna :
simulations in GLAMER

Contributors : Ben Metcalf (PI), Carlo Giocoli, Dominik Leier, Fabio Bellagamba, Alkistis Pourtsidou, Me (postdocs), M. Petkova (former), Nicolas Tessore, Alessandro Romeo (PhDs).

Coding on big project

The GLAMER code

GLAMER = **G**ravitational **L**ensing code with **A**daptive **Me**sh **R**efinement.

The code can simulate a large variety of **sources** :

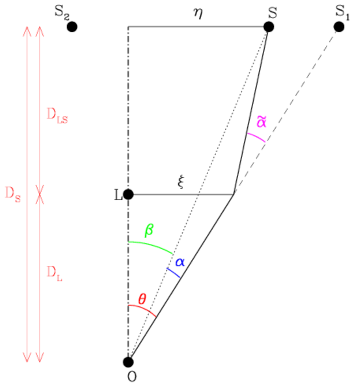
- simple sources : point source, uniform distribution, Gaussian,
- quasars and their emission region,
- galaxy models,
- array of pixels (i.e. any image), shapelets, ...

and **lenses** :

- analytic halo models : point masses, SIS, NFW, NSIE, Jaffe, Hernquist,...
- smoothed simulation particles,
- pixel maps (i.e. MOKA maps),

and shoot rays in an **adaptive** way over large angles (several deg^2), refining a **grid** to reach the desired precision down to microlensing scales ($\sim 10^{-4}$pc).

Ray shooting :



Lens equation : $\boldsymbol{\eta} = \frac{D_s}{D_l}\boldsymbol{\xi} - D_{ls}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})$

Dimensionless quantities :
$\boldsymbol{y} = \boldsymbol{\eta}/\eta_0$, $\mathbf{x} = \boldsymbol{\xi}/\xi_0$, $\eta_0 = \frac{D_s}{D_l}\xi_0$

$$\Rightarrow \quad \boxed{\boldsymbol{y} = \mathbf{x} - \frac{D_{ls}D_l}{D_s}\hat{\boldsymbol{\alpha}}\left(\boldsymbol{\xi}\right) \equiv \mathbf{x} - \boldsymbol{\alpha}}$$

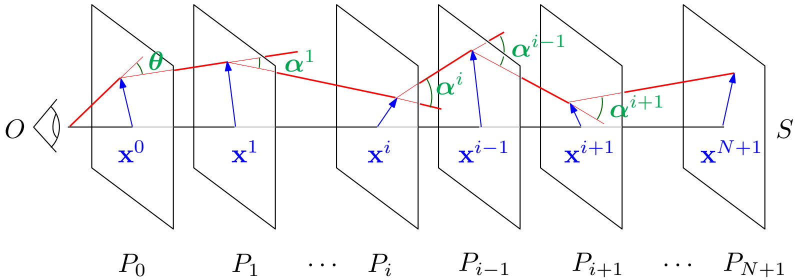
Amplification matrix :

$$\mathcal{A} = \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\mathbf{x}} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2+\omega \\ -\gamma_2-\omega & 1-\kappa+\gamma_1 \end{pmatrix}$$

- $\kappa = \frac{1}{2}\left(\partial_1\alpha_1 + \partial_2\alpha_2\right)$ (convergence),
- $\gamma_1 = \frac{1}{2}\left(\partial_1\alpha_1 - \partial_2\alpha_2\right)$ (shear),
- $\gamma_2 = \partial_1\alpha_2 = \partial_2\alpha_1$ (shear),
- $\omega = \frac{1}{2}\left(\partial_1\alpha_2 - \partial_2\alpha_1\right)$ (vorticity).

We also define the **lensing potential** ϕ : $\boldsymbol{\alpha} = \nabla\phi$ such that $2\kappa = \nabla^2\phi$.

Multiplane lens equation



The **multiplane lens equation** is given by :

$$\mathbf{x}^{i+1} = \mathbf{x}^{i} - D_{i+1,i}\Big(\boldsymbol{\theta} + \sum_{j=1}^{i} \boldsymbol{\alpha}^{j}(\mathbf{x}^{k})\Big) \quad ,$$

with I.C. : $\mathbf{x}^0 = 0$ and $\mathbf{x}^1 = \boldsymbol{\theta}\, D_1$, which by recursion gives :

$$\mathbf{x}^{i+1} = \left(\frac{D_{i+1,i}}{D_{i,i-1}} + 1\right)\mathbf{x}^{i} - \frac{D_{i+1,i}}{D_{i,i-1}}\mathbf{x}^{i-1} - D_{i+1,i}\,\boldsymbol{\alpha}^{i}(\mathbf{x}^{j}) \quad .$$

Deriving lensing quantities

We define an **amplification matrix** for each plane : $\boldsymbol{A}^i \equiv \frac{1}{D_i}\frac{\partial \mathbf{x}^i}{\partial \boldsymbol{\theta}}$. We get :

$$\boldsymbol{A}^{i+1} = \frac{D_{i+1,i-1}}{D_{i,i-1}}\frac{D_i}{D_{i+1}}\boldsymbol{A}^i - \frac{D_{i+1,i}}{D_{i,i-1}}\frac{D_{i-1}}{D_{i+1}}\boldsymbol{A}^{i-1} - \frac{D_{i+1,i}D_i}{D_{i+1}}\boldsymbol{G}^i\,\boldsymbol{A}^i \quad ,$$

with I.C. : $\boldsymbol{A}^0 = 0$ and $\boldsymbol{A}^1 = \boldsymbol{I}_{\mathrm{d}}$.

$\boldsymbol{G}^i$ is a forcing term such that :

$$\boldsymbol{G}^i \equiv \frac{\partial \boldsymbol{\alpha}^i}{\partial \mathbf{x}^i} = (\kappa)^i\,\boldsymbol{I}_{\mathrm{d}} + (\gamma_1)^i\,\boldsymbol{\sigma}_1 + (\gamma_2)^i\,\boldsymbol{\sigma}_2 \quad ,$$

when $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ are symmetric matrices : $\boldsymbol{\sigma}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Consequences on the code, we can :

- do ray tracing through **different lens planes** (with grid refinement),
- work with **different types of halos** (putting them on a tree).

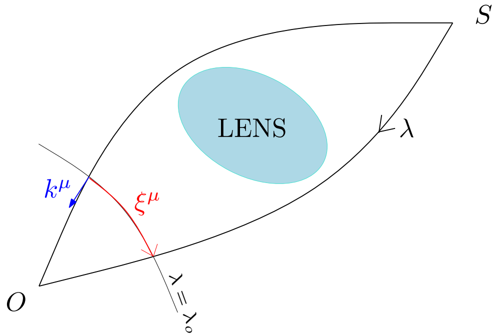
The Jacobi Map

Geodesic deviation equation

We consider 2 light rays emitted at the same time from a **source** S and converging to an **observer** O. $\forall$ time, their relative separation follows :

$$\boxed{\nabla^2_\lambda \xi^\mu = R^\mu_{\alpha\beta\nu} k^\alpha k^\nu \xi^\beta \quad , \quad \nabla_\lambda \equiv \mathrm{D}/\mathrm{d}\lambda \equiv k^\mu \nabla_\mu}$$

where k^μ is the photon momentum, $\nabla_\lambda \equiv k^\mu \nabla_\mu$ with λ an affine parameter along the photon path, and ξ^μ **an $\perp$ displacement** wrt to the LoS.



Jacobi equation

The geodesic equation is projected thanks to the **Sachs basis** $\{s^\mu_A\}_{A=1,2}$:

$$g_{\mu\nu} s^\mu_A s^\nu_B = \delta_{AB} \quad , \quad s^\mu_A u_\mu = 0 \quad , \quad s^\mu_A k_\mu = 0 \quad , \quad \Pi^\mu_\nu \nabla_\lambda s^\nu_A = 0$$

- u_μ is the **peculiar velocity** of the comoving fluid (and S and O),
- $\Pi^\mu_\nu = \delta^\mu_\nu - \frac{k^\mu k_\nu}{(u^\alpha k_\alpha)^2} - \frac{k^\mu u_\nu + u^\mu k_\nu}{u^\alpha k_\alpha}$ a projector on the **screen**,
- the **screen** is orthogonal to u_μ and $n_\mu \equiv u_\mu + (u^\alpha k_\alpha)^{-1} k_\mu$.

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We define the **Jacobi map** $(\bar{\theta}^A_o \to \xi^A = \xi^\mu s^A_\mu)$:

$$\boxed{\xi^A(\lambda) = J^A_B(\lambda, \lambda_o) \bar{\theta}^A_o}$$

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$\Rightarrow$ the **projected quantities** ξ^A and $R^A_B = R_{\alpha\beta\nu\mu} k^\alpha k^\nu s^\beta_B s^\mu_A$ (optical tidal matrix) bring us the (linear) 2nd order differential **Jacobi equation** :

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} J^A_B(\lambda, \lambda_o) = R^A_C(\lambda)\, J^C_B(\lambda, \lambda_o) \quad ,}$$

$$\text{with I.C. :} \qquad J^A_B(\lambda_o, \lambda_o) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} J^A_B(\lambda_o, \lambda_o) = (k^\mu u_\mu)_o\, \delta^A_B \quad ,$$

Remark : See e.g. Bonvin etal '11, Fleury etal '13, Fanizza etal '13.

Physical quantities

Angular distance of the source: $d_A(\lambda_s) \equiv \frac{dS_s}{d^2\Omega_o} = \sqrt{\det J^A_B(\lambda_s, \lambda_o)}$.

The (unlensed) angular position of the source $\bar{\theta}^A_s$ and the observed lensed position $\bar{\theta}^A_o$ (of the image) are:

$$\bar{\theta}^A_s = \left(\frac{\xi^A}{\bar{d}_A}\right)_s \quad , \quad \bar{\theta}^A_o = \left(\frac{k^\mu \partial_\mu \xi^A}{k^\mu u_\mu}\right)_o \quad .$$

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The **amplification matrix** is defined as :

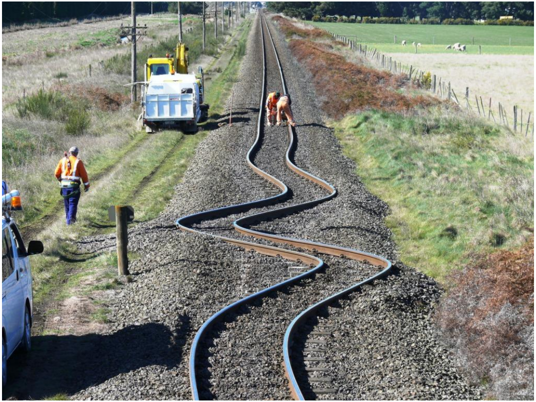
$$\boxed{\mathcal{A}^A_B \equiv \frac{\mathrm{d}\bar{\theta}^A_s}{\mathrm{d}\bar{\theta}^B_o} = \frac{J^A_B(\lambda_s, \lambda_o)}{\bar{d}_A(\lambda_s)} = \begin{pmatrix} 1 - \kappa - \hat{\gamma}_1 & -\hat{\gamma}_2 + \hat{\omega} \\ -\hat{\gamma}_2 - \hat{\omega} & 1 - \kappa + \hat{\gamma}_1 \end{pmatrix}}$$

Hence the **lensing quantities** are :

$$\kappa = 1 - \frac{\mathrm{tr} J^A_B}{2\bar{d}_A} \quad , \quad \mu = (\det \mathcal{A})^{-1} = \frac{\bar{d}^2_A}{\det J^A_B} \quad , \quad \hat{\omega} = \frac{|J^1_2 - J^2_1|}{2\bar{d}_A} \quad ,$$

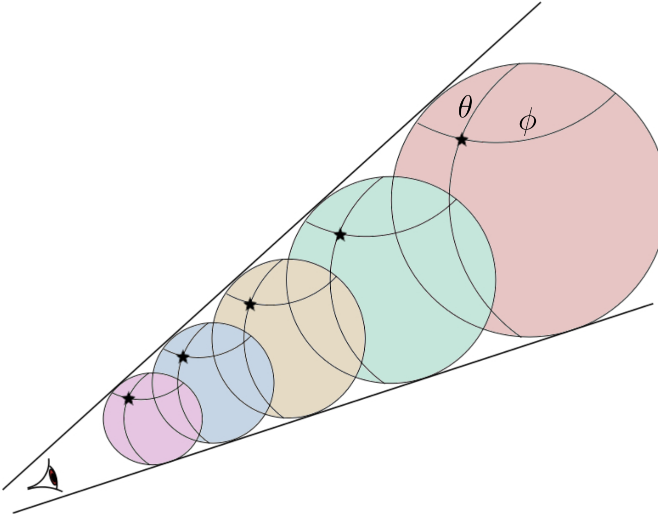
$$|\hat{\gamma}|^2 = (1-\kappa)^2 + \hat{\omega}^2 - \mu^{-1} = \left(\frac{\mathrm{tr} J^A_B}{2\bar{d}_A}\right)^2 + \left(\frac{|J^1_2 - J^2_1|}{2\bar{d}_A}\right)^2 - \frac{\det J^A_B}{\bar{d}^2_A} \quad ,$$

Inhomogeneous coordinates



This railway is not wrong, it is just adapted to shorter trains :).

(In-)Homogeneous lightcone observations :



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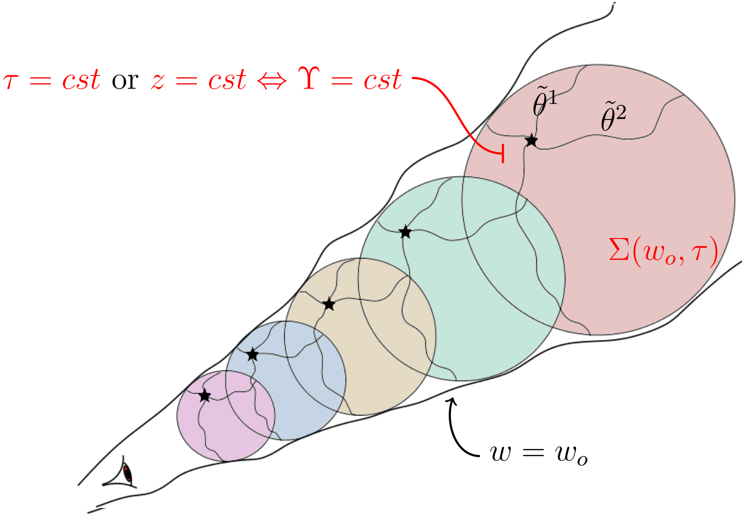

Geodesic light-cone coordinates

A light-cone adapted metric (close to "*observational coordinates*") :

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)$$

(6 arbitrary functions : Υ, U^a, γ_{ab})

Properties :

- w is a null coordinate : $\partial_\mu w \, \partial^\mu w = 0$,
- $\partial_\mu \tau$ defines a geodesic flow : $(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) \equiv 0$ (from $g^{\tau\tau}_{GLC} = -1$) ,
- an observer defined by constant τ spacelike hyp. is in geodesic motion,
- photons travel at $(w, \tilde{\theta}^a) = \overrightarrow{cst}$ and their path is orthogonal to $\Sigma(w, z)$.

Interpretation : Υ is like an **inhomogeneous scale factor** (lapse function), U^a like a shift-vector and γ_{ab} **the metric inside the 2-sphere**.

FLRW : $w = \eta + r$, $\tau = t$ (exact if t = synchronous gauge time)

$(\tilde{\theta}^1, \tilde{\theta}^2) = (\theta, \phi)$, $\gamma_{ab} d\tilde{\theta}^a d\tilde{\theta}^b = a^2 r^2 d^2\Omega$,

$\Upsilon = a(t)$, $U^a = 0$.

It makes life easier !

$$\mathrm{d}s^2_{GLC} = \Upsilon^2 \mathrm{d}w^2 - 2\Upsilon \mathrm{d}w \mathrm{d}\tau + \gamma_{ab}(\mathrm{d}\widetilde{\theta}^a - U^a \mathrm{d}w)(\mathrm{d}\widetilde{\theta}^b - U^b \mathrm{d}w)$$

$\Rightarrow$ **Redshift perturbation**:

$$(1+z_s) = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{(\partial^\mu w \partial_\mu \tau)_s}{(\partial^\mu w \partial_\mu \tau)_o} = \frac{\Upsilon(w_o, \tau_o, \widetilde{\theta}^a)}{\Upsilon(w_o, \tau_s, \widetilde{\theta}^a)} \equiv \frac{\Upsilon_o}{\Upsilon_s}$$

where $u_\mu = -\partial_\mu \tau$ is the **peculiar velocity** of the comoving observer/source and $k_\mu = \partial_\mu w$ is the **photon momentum**.

$\Rightarrow$ **(exact) Angular distance** (homogeneous observer neighborhood):

$$d_A = \gamma^{1/4} \left(\sin \tilde{\theta}^1\right)^{-1/2} \quad \text{with} \quad \gamma \equiv \det(\gamma_{ab}) = |\det(g_{\mathrm{GLC}})|/\Upsilon^2$$

which, combined with the redshift, gives the distance-redshift relation.

Examples of applications

- Compute the **distance-redshift relation** at $\mathcal{O}(2)$ in perturbations (from the Newtonian gauge, 1104.1167) :

$$d_L(z_s,\theta^a) = d_L^{FLRW}(z_s)\left(1+\delta_S^{(1)}(z_s,\theta^a)+\delta_S^{(2)}(z_s,\theta^a)\right)$$

- Simplify **averages on the past lightcone** (1207.1286, 1302.0740) :

$$\begin{aligned}\langle S\rangle_{w_o,\tau_s} &= \frac{\int_\Sigma d^4x\sqrt{-g}\,\delta_{\rm D}(w-w_o)\delta_{\rm D}(\tau-\tau_s)\,|\partial_\mu\tau\partial^\mu w|\,S(\tau,w,\widetilde{\theta}^a)}{\int_\Sigma d^4x\sqrt{-g}\,\delta_{\rm D}(w-w_o)\delta_{\rm D}(\tau-\tau_s)\,|\partial_\mu\tau\partial^\mu w|} \\ &= \left(\int d^2\widetilde{\theta}\sqrt{\gamma(w_o,\tau_s,\widetilde{\theta}^a)}\,S(w_o,\tau_s,\widetilde{\theta}^a)\right)\Big/\left(\int d^2\widetilde{\theta}\sqrt{\gamma(w_o,\tau_s,\widetilde{\theta}^a)}\right) \quad ,\end{aligned}$$

- Estimate the effect of the large scale structure on the **Hubble diagram** : average and dispersion of the distance modulus (my thesis, 1309.6542).

- Evaluate the galaxy number counts at $\mathcal{O}(2)$ in perturbations (Di Dio, Durrer, Marozzi, Montanari 1407.0376v3).

Jacobi in GLC coordinates

The **zweibeins** are written as $s^\mu_A = (s^\tau_A, 0, s^a_A)$ and $k^\mu = \omega \Upsilon^{-1} \delta^\mu_\tau$.

The **solution to the Jacobi equation** (and its I.C.s) is :

$$\boxed{J^A_B(\lambda, \lambda_o) = s^A_a(\lambda) \left[2u_\tau (\dot{\gamma}_{ab})^{-1}\right]_o s^B_b(\lambda_o) \equiv s^A_a(\lambda) \Delta^{ab}(\lambda_o) s^B_b(\lambda_o)}$$

where $(\ldots)^{\bullet} \equiv \partial_\tau(\ldots)$.

The **angular distance** and the **magnification** become :

$$\boxed{d_A = \frac{2u_{\tau_o}}{\sqrt{[\det^{ab} \dot{\gamma}_{ab}]_o}} (\gamma\gamma_o)^{1/4}}$$

$$\mu = \frac{u_{\tau_o}^{-2} \bar{d}_A^2}{4\sqrt{\gamma\gamma_o}} \left[\det{}^{ab} \dot{\gamma}_{ab}\right]_o \equiv \left(\frac{\bar{d}_A}{d_A}\right)^2 \quad ,$$

involving $\bar{d}_A = a^2(\tau) r^2$ with $r = w - \int a^{-1}(\tau) \mathrm{d}\tau$ measured from the observer.

Lensing quantities in GLC coordinates

We compute the squared quantities (more convenient) and use $\Delta_o^{ab} \equiv \Delta^{ab}(\lambda_o)$:

$$
\begin{aligned}
(1-\kappa)^2 &= \frac{1}{4\bar{d_A}^2}\left(J_1^1 + J_2^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1 \left(s_b^1 s_d^1\right)_o + s_a^2 s_c^2 \left(s_b^2 s_d^2\right)_o + 2 s_a^1 s_c^2 \left(s_b^1 s_d^2\right)_o\right] \\
\hat{\omega}^2 &= \frac{1}{4\bar{d_A}^2}\left(J_2^1 - J_1^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1 \left(s_b^2 s_d^2\right)_o + s_a^2 s_c^2 \left(s_b^1 s_d^1\right)_o - 2 s_a^1 s_c^2 \left(s_b^2 s_d^1\right)_o\right] \\
\hat{\gamma}_1^2 &= \frac{1}{4\bar{d_A}^2}\left(J_1^1 - J_2^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1 \left(s_b^1 s_d^1\right)_o + s_a^2 s_c^2 \left(s_b^2 s_d^2\right)_o - 2 s_a^1 s_c^2 \left(s_b^1 s_d^2\right)_o\right] \\
\hat{\gamma}_2^2 &= \frac{1}{4\bar{d_A}^2}\left(J_2^1 + J_1^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1 \left(s_b^2 s_d^2\right)_o + s_a^2 s_c^2 \left(s_b^1 s_d^1\right)_o + 2 s_a^1 s_c^2 \left(s_b^2 s_d^1\right)_o\right]
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\hat{\omega}^2 &= \frac{1}{4\bar{d_A}^2}\left(J_2^1-J_1^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1\left(s_b^2 s_d^2\right)_o + s_a^2 s_c^2\left(s_b^1 s_d^1\right)_o - 2 s_a^1 s_c^2\left(s_b^2 s_d^1\right)_o\right] \\
\hat{\gamma}_1^2 &= \frac{1}{4\bar{d_A}^2}\left(J_1^1-J_2^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1\left(s_b^1 s_d^1\right)_o + s_a^2 s_c^2\left(s_b^2 s_d^2\right)_o - 2 s_a^1 s_c^2\left(s_b^1 s_d^2\right)_o\right] \\
\hat{\gamma}_2^2 &= \frac{1}{4\bar{d_A}^2}\left(J_2^1+J_1^2\right)^2 = \frac{1}{4\bar{d_A}^2}\Delta_o^{ab}\Delta_o^{cd}\left[s_a^1 s_c^1\left(s_b^2 s_d^2\right)_o + s_a^2 s_c^2\left(s_b^1 s_d^1\right)_o + 2 s_a^1 s_c^2\left(s_b^2 s_d^1\right)_o\right]
\end{aligned}
$$

which can be simplified thanks to $s_a^A s_b^A = \gamma_{ab}$ and $\epsilon_{AB}\, s_a^A s_b^B = \sqrt{\gamma}\,\epsilon_{ab}$ into :

$$
\left\{ \begin{array}{c} (1-\kappa)^2+\hat{\omega}^2 \\ \hat{\gamma}_1^2+\hat{\gamma}_2^2 \end{array} \right\} = \left(\frac{u_{\tau_o}}{\bar{d}_A}\right)^2 \left\{ \left[\frac{\gamma\,\dot{\gamma}_{ab}\gamma^{bc}\dot{\gamma}_{cd}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)^2}\right]_o \gamma\,\gamma^{ad} \pm 2\frac{\sqrt{\gamma\,\gamma_o}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)_o} \right\}
$$

Similar expressions for the deformation matrix and the optical scalars.

Deformation matrix and optical scalars

The Jacobi equation can be rewritten :

$$\frac{\mathrm{d}J^A_B}{\mathrm{d}\lambda} = \mathcal{S}^A_C J^C_B \quad , \quad \frac{\mathrm{d}\mathcal{S}^A_B}{\mathrm{d}\lambda} + \mathcal{S}^A_C \mathcal{S}^C_B = R^A_B \quad ,$$

where the **deformation matrix** is :

$$\boxed{\mathcal{S}^A_B \equiv \frac{\mathrm{d}J^A_C}{\mathrm{d}\lambda}(J^{-1})^C_B = \hat{\theta}\,\delta^A_B + \begin{pmatrix} \hat{\sigma}_1 & \hat{\sigma}_2 \\ \hat{\sigma}_2 & -\hat{\sigma}_1 \end{pmatrix}}$$

The second equation at the top becomes the **Sachs equations** $(\hat{\sigma} \equiv \hat{\sigma}_1 + i\hat{\sigma}_2)$:

$$\tfrac{\mathrm{d}\hat{\theta}}{\mathrm{d}\lambda} + |\hat{\sigma}|^2 + \hat{\theta}^2 = \tfrac{1}{2}\mathrm{tr}R^A_B \equiv \Phi_{00} \qquad , \qquad \tfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\lambda} + 2\hat{\theta}\hat{\sigma} \equiv \Psi_0 \quad .$$

The **optical scalars** can be obtained by solving these equations or directly :

$$\hat{\theta} \equiv \frac{1}{2}\nabla_\mu k^\mu \quad \text{(expansion scalar)} \quad ,$$

$$|\hat{\sigma}|^2 \equiv \frac{1}{2}\nabla_\mu k_\nu \nabla^\mu k^\nu - \hat{\theta}^2 \quad \text{(shear scalar)} \quad .$$

Ricci and Weyl focusing

Riemann and the Weyl tensors are related by :

$$C_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} - g_{\alpha[\mu}R_{\nu]\beta} + g_{\beta[\mu}R_{\nu]\alpha} + \frac{1}{3}R\, g_{\alpha[\mu}g_{\nu]\beta} \quad .$$

The optical tidal matrix can be decomposed as :

$$R^A_B = \Phi_{00}\,\delta^A_B + \begin{pmatrix} \mathrm{Re}\Psi_0 & \mathrm{Im}\Psi_0 \\ \mathrm{Im}\Psi_0 & -\mathrm{Re}\Psi_0 \end{pmatrix}$$

and we can show that the **Ricci focusing** and **Weyl focusing** are equal to :

$$\Phi_{00} = -\frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta \quad , \quad \Psi_0 = \frac{1}{2}C_{\alpha\beta\mu\nu}k^\alpha k^\mu \Sigma^\beta \Sigma^\nu \quad ,$$

where $\Sigma^\mu \equiv s_1^\mu + i s_2^\mu$.

Thanks to the Einstein equations, we can directly link Φ_{00} to the matter content inside the beam. Ψ_0 is related to the matter content outside the beam.

Deformation matrix in GLC coordinates :

The **deformation matrix** is given in terms of the zweibeins by :

$$\boxed{\mathcal{S}^A_B = \frac{\mathrm{d}s^A_a}{\mathrm{d}\lambda} s^a_B = \frac{\omega}{2\Upsilon} s^a_A s^b_B \, \dot{\gamma}_{ab}}$$

Using $s^a_A s^b_A = \gamma^{ab}$ we get the optical scalars :

$$\boxed{\hat{\theta} = \omega \, \frac{\gamma^{ab}\dot{\gamma}_{ab}}{4\Upsilon} = \frac{\omega}{4\Upsilon}\frac{\dot{\gamma}}{\gamma} \quad , \quad |\hat{\sigma}|^2 = \left(\frac{\omega}{4\Upsilon}\frac{\dot{\gamma}}{\gamma}\right)^2 - \frac{\omega^2}{4\Upsilon^2}\,\frac{\det \dot{\gamma}_{ab}}{\gamma}}$$

We also get the **Ricci and Weyl focusing** in the GLC gauge :

$$\Phi_{00} = \frac{\omega^2}{4\Upsilon^2}\left[\gamma^{ab}\ddot{\gamma}_{ab} - \frac{\dot{\Upsilon}}{\Upsilon}\gamma^{ab}\dot{\gamma}_{ab} - \frac{1}{2}\gamma^{ab}\dot{\gamma}_{ac}\gamma^{cd}\dot{\gamma}_{db}\right] \quad ,$$

$$|\Psi_0|^2 = \frac{\omega^4}{16\Upsilon^4}\left[\ddot{\gamma}_{ab} - \frac{\dot{\Upsilon}}{\Upsilon}\dot{\gamma}_{ab} - \frac{1}{2}\dot{\gamma}_{ae}\gamma^{ef}\dot{\gamma}_{fb}\right]\left[\ddot{\gamma}_{cd} - \frac{\dot{\Upsilon}}{\Upsilon}\dot{\gamma}_{cd} - \frac{1}{2}\dot{\gamma}_{cg}\gamma^{gh}\dot{\gamma}_{hd}\right]\left(\gamma^{ac}\gamma^{bd} + \gamma^{ad}\gamma^{bc} - \gamma^{ab}\gamma^{cd}\right) \, .$$

Application to an inhomogeneous model : off-center observer in a LTB model.

Off-center observer in LTB

The **Lemaître-Tolman-Bondi** (LTB) coord. are defined by the line element :

$$\mathrm{d}s^2_{\mathrm{LTB}} = -\mathrm{d}t^2 + X^2(t,r)\mathrm{d}r^2 + A^2(t,r)\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right] \quad .$$

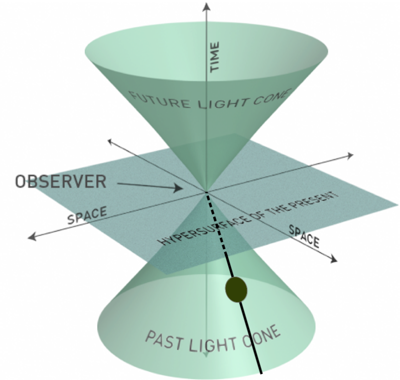
An observer at $r = 0$ sees an isotropic Universe around him, but any other at $r = d$ from the center sees an anisotropy.

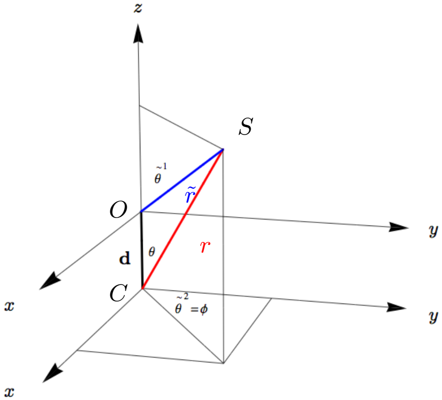
The **transformation** of coordinates between **LTB** and **GLC** gives :

$$\begin{aligned} \tau &= t \quad \text{(thanks to the synchronous form of the LTB metric)} \quad , \\ w &= W(t,r,\theta) \quad , \\ \tilde{\theta}^1 &= \arccos\left(\frac{r\cos\theta - d}{\sqrt{r^2 + d^2 - 2rd\cos\theta}}\right) \quad , \\ \tilde{\theta}^2 &= \phi \quad \text{(by assumption)} \quad , \end{aligned}$$

$$\Upsilon = \frac{1}{\partial_t W} \quad , \quad U^a = \vec{0} \quad , \quad \gamma^{ab} = \begin{pmatrix} \frac{A^2\,d^2\sin^2\theta + r^2X^2(r-d\cos\theta)^2}{A^2X^2\left(d^2+r^2-2rd\cos\theta\right)^2} & 0 \\ 0 & A^{-2}\sin^{-2}\theta \end{pmatrix} \quad .$$

Illustration of LTB





In GLC coordinates (3+1)

(x, y, z) between GLC and LTB

Expressing $\sqrt{\gamma}$ and $\left(\frac{4\sqrt{\gamma}}{\det^{ab}\dot{\gamma}_{ab}}\right)_o$ at the observer position (t,r,θ,ϕ)=$(t_o,d,0,0)$, the **angular distance** becomes :

$$d_A^2 = \frac{A^2 X\left(r^2+d^2-2\,rd\cos\theta\right)}{\sqrt{A^2\,d^2\sin^2\theta + r^2X^2\left(r-d\cos\theta\right)^2}}\,\frac{A_0(d)}{d\,X_0(d)}\,\frac{\sin\theta}{\sin\tilde{\theta}} \quad .$$

In the LTB metric we can always manage a residual gauge degree of freedom in order to fix $A(t_o,r)\equiv A_0(r)=r$. Moreover, in the flat case, using the off-diagonal Einstein equations, we can write $X(t,r)=\partial_r A(t,r)$.

In the **flat FLRW** case we get $A(t,r)\to r\,a(t)$ and $X(t,r)\to a(t)$ and as expected :

$$\bar{d}_A^2 = \frac{a^2\sqrt{r^2+d^2-2\,rd\cos\theta}\;r\sin\theta}{\sin\tilde{\theta}} \equiv \tilde{r}^2 a(t)^2 \quad .$$

with $\tilde{r}=\sqrt{r^2+d^2-2\,rd\cos\theta}$.

$k(r)=0 \Rightarrow$ we have only the **decaying mode**.

Here γ_{ab} is diagonal $\Rightarrow \hat{\omega} = 0$, i.e. no vorticity.

The other **lensing quantities** (using $u_{\tau_o} = 1$ and $\bar{d}_A$) are:

$$\begin{aligned}
\mu &= \frac{r\, d\, X_0(d)\, a^2(t)}{A_0(d)\, A^2(t,r)\, X(t,r)} \sqrt{\frac{A^2(t,r)\, d^2 \sin^2\theta + r^2 X^2(t,r)\,(r - d\cos\theta)^2}{d^2 + r^2 - 2rd\cos\theta}} \quad , \\
(1-\kappa)^2 &= \frac{A^2(t,r)}{4\sin\theta\, d^2 r\, a^2(t) X_0^2(d) \sqrt{d^2 - 2dr\cos\theta + r^2}} \Big[A + B + C \Big] \quad , \\
|\hat{\gamma}|^2 &= \frac{A^2(t,r)}{4\sin\theta\, d^2 r\, a^2(t) X_0^2(d) \sqrt{d^2 - 2dr\cos\theta + r^2}} \Big[A + B - C \Big] \quad ,
\end{aligned}$$

with:

$$\begin{aligned}
A &= d^2 \sin^2\theta X_0^2(d) \quad , \\
B &= \frac{A_0^2(d) X^2(t,r) \left(d^2 - 2dr\cos\theta + r^2\right)^2}{d^2 \sin^2\theta A^2(t,r) + r^2 X^2(t,r)(r - d\cos\theta)^2} \quad , \\
C &= \frac{2d\sin\theta A_0(d) X_0(d) X(t,r) \left(d^2 - 2dr\cos\theta + r^2\right)}{\sqrt{d^2 \sin^2\theta A^2(t,r) + r^2 X^2(t,r)(r - d\cos\theta)^2}} \quad .
\end{aligned}$$

... not so elegant :(.

For a collection of non interacting perfect barotropic fluids, we have:

$$H^2(t,r) = H_0^2(r) \sum_n \Omega_{n0}(r) \left[\frac{A_0(r)}{A(t,r)}\right]^{\alpha_n} \quad , \quad \sum_n \Omega_{n0}(r) = 1 \quad , \quad X(t,r) = \frac{\partial_r A(t,r)}{\sqrt{1-k(r)}} \quad ,$$

where:

- $H(t,r) \equiv \partial_t A(t,r)/A(t,r)$,
- $H_0(r) \equiv H(t_0,r)$ is the inhomogeneous Hubble function "today",
- $k(r)$ is the inhomogeneous spatial curvature (for simplicity $k(r)=0$ here).

CDM case: $\Omega_{m0}(r) = 1$

ΛCDM case: $\Omega_{m0}(r) = 1 - \Omega_{\Lambda 0}(r) = 1 - \Omega_{\Lambda 0} \left(\frac{H_0}{H_0(r)}\right)^2$

$\Omega_{\Lambda 0}$, H_0: values of the homogeneous case taken as background quantities:
$H_0 \equiv \lim_{r\to\infty} H_0(r)$ and $\Omega_{\Lambda 0}$ such that $H_0^2(r)\Omega_{\Lambda 0}(r) = \Omega_{\Lambda 0} H_0^2 \equiv \Lambda/3$.

$\Rightarrow H_0(r)$ **completely takes into account the density profile of matter** and so, by choosing it, we can directly study the under/overdensity we want.

Ansatz for $H_0(r)$:

$$H_0(r) = H_0 \sqrt{1 - \frac{H_0^2 - H_{\rm in}^2}{H_0^2} \frac{\tanh\left(\frac{d-r_0}{2\,\Delta r}\right) - \tanh\left(\frac{r-r_0}{2\,\Delta r}\right)}{\tanh\left(\frac{d-r_0}{2\,\Delta r}\right) + \tanh\left(\frac{r_0}{2\,\Delta r}\right)}}$$

where :

- r_0 is the radius of the under/overdensity ($d \gg r_0$),
- Δr is the transition scale from bubble to background ($\Delta r \ll r_o \ll d$),
- $H_0 = 70$ km s^{-1} Mpc^{-1}, and $H_{in} = H_0(r = 0)$.
- Numerically : $r_0 = 1$ Mpc, $\Delta r = 0.1$ Mpc, $|H_0 - H_{\rm in}| = 2$ km s^{-1} Mpc^{-1}.

The **time t** between today and the big bang is :

$$t_0 - t = \int_{A(t,r)}^{A_0(r)} \frac{\mathrm{d}A}{AH(t,r)} = \frac{1}{H_0(r)} \int_{A(t,r)/A_0(r)}^{1} \frac{\mathrm{d}x}{x\sqrt{\Omega_{m0}(r)x^{-3} + \Omega_{\Lambda 0}(r)}} \quad ,$$

$\Rightarrow$ can inverse this relation to get $A(t,r)$ and we set $A_0(r) = r$.

The inversion gives the following expansion factor :

Inhomogeneous CDM model : $(\Omega_{m0}(r) = 1 \quad , \quad t_0 = 0)$

$$A(t,r) = r\left[1 + \frac{3}{2}\,H_0(r)\,t\right]^{2/3}$$

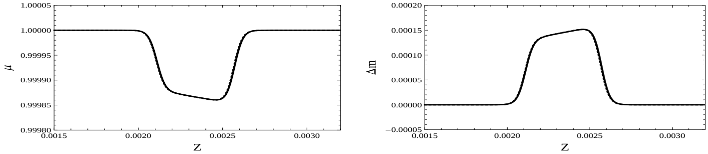
Inhomogeneous ΛCDM model :
$(\Omega_{\Lambda 0}(r) + \Omega_{m0}(r) = 1 \quad , \quad H_0^2 \quad , \quad \Omega_{\Lambda 0} = H_0^2(r)\,\Omega_{\Lambda 0}(r) \quad , \quad t_0 = 0)$

$$A(t,r) = r\left[\frac{1-\Omega_{\Lambda 0}(r)}{\Omega_{\Lambda 0}(r)}\right]^{1/3}\left(\sinh\left[\operatorname{arcsinh}\sqrt{\frac{\Omega_{\Lambda 0}(r)}{1-\Omega_{\Lambda 0}(r)}} + \frac{3}{2}\sqrt{\Omega_{\Lambda 0}(r)}\,H_0(r)\,t\right]\right)^{2/3}$$

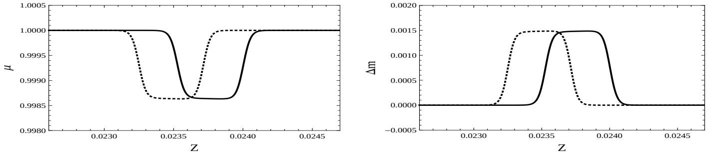
Background values : $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{\Lambda 0} = 0.68$.

Illustration

Under/overdensity at $d = 10$ or 100 Mpc from the observer, with a radius $r_0 = 1$ Mpc, a transition shell of $\Delta r = 0.1$ Mpc, and $|H_0 - H_{\text{in}}| = 2\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$.



(a) Underdensity at d=10 Mpc

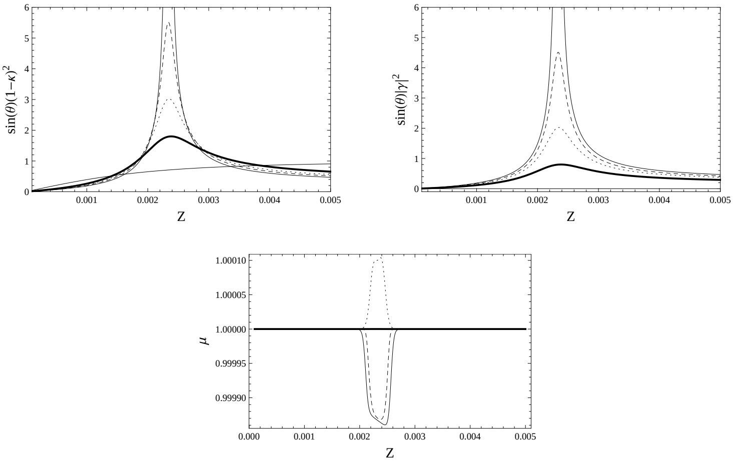


(b) Underdensity at d=100 Mpc

Left side : magnification ; right side : distance modulus $\Delta m = 5\,\log_{10}(d_A/\bar{d}_A)$.
Solid lines : CDM model ; dotted lines : ΛCDM model.

Source not aligned with bubble

Plot of μ, $\sin\theta(1-\kappa)^2$ and $\sin\theta|\hat{\gamma}|^2$ at different values of the angle $\tilde{\theta}$, namely $\tilde{\theta} = \pi - \arcsin(\{10, 2, 1, 0.5, 0\} \times r_o/d)$



Underdensity with $r_o = 1\,$Mpc, $d = 10\,$Mpc in ΛCDM.
Angles of observation: $\tilde{\theta} = \pi - \arcsin(10\, r_o/d)$ (thin), $\pi - \arcsin(2\, r_o/d)$ (thick), $\pi - \arcsin(r_o/d)$ (dotted) and $\pi - \arcsin(r_o/2d)$ (dashed), π (thin).

Remarks

To obtain $r(z)$ we **integrated numerically** the geodesic equation in LTB (Blomqvist, Mortsell 2010) :

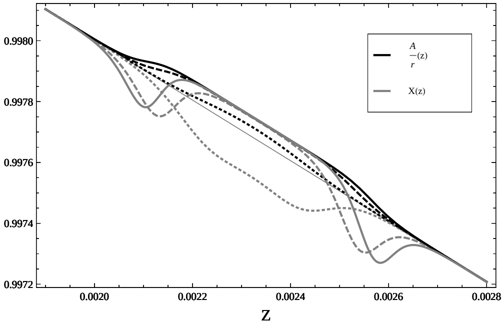
$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{(1+z)}{q} \quad , \quad \frac{\mathrm{d}r}{\mathrm{d}z} = \frac{p}{q} \quad , \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{J}{qA^2} \quad ,$$
$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{1}{q}\left[\frac{(1-k)}{A'}\frac{J^2}{A^3} + \frac{2\dot{A}'}{A'}p(1+z) - \left(\frac{A''}{A'} + \frac{k'}{2-2k}\right)p^2\right] \quad ,$$

with the constraint $q = \left[\frac{A'\dot{A}'}{1-k}p^2 + \frac{\dot{A}J^2}{A^3}\right]$ and $p = \mathrm{d}r/\mathrm{d}\lambda$, $A' \equiv \partial_r A$, $\dot{A} \equiv \partial_t A$.
$J = A_0(d)\sin\tilde{\theta}$ is a constant angular momentum
We use the I.C. : $t = 0$, $r = d$, $\theta = 0$, $p = \cos\tilde{\theta}/A_0'(r)$.

Important : This necessity of solving the geodesic equation in LTB is due to the “unobservable” aspect of the LTB coordinates (t, r) compared to the GLC ones (where e.g. Υ is directly related to z).

We considered an uncompensated LTB under/overdensity model with $\mathbf{k(r) = 0}$ (i.e. only the decaying mode) ⇒ possibility of **generalisation**.

$A(t,r)/r$ **and** $X(t,r)$ **do not diverge** (see figure)
$\Rightarrow$ our metric functions are free from singularities.



Underdensity with $r_o = 1\,\mathrm{Mpc}$ situated at $d = 10\,\mathrm{Mpc}$ in a ΛCDM background.
Angles of observation: $\pi - \arcsin(10\, r_o/d)$ (thin gray), $\pi - \arcsin(r_o/d)$ (dotted) and $\pi - \arcsin(r_o/2d)$ (dashed), $\tilde{\theta} = \pi$ (thin).

Conclusions

Advantages of using the GLC coordinates :

- They are **adapted** to calculations involving **light-propagation**,
- They can also be useful for **weak lensing** (where γ_{ab} is like a screen),
- It may help to get **new predictions** (?),
- **Other aspects** of lensing could be studied (e.g. lensing statistics) (?).

On the other hand :

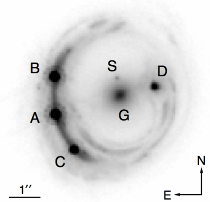
- It is probably **not adapted** to problems with timelike propagation,
- Writing **Einstein equations** is not so easy in this system of coordinates,
- **Transformation of coordinates** with GLC is not always that easy.


Thank you!

Appendices

(details on previous slides)

Time delays



Time delay between different images :

$$\Delta t_{ij} = \frac{D_{\Delta t}}{c} \left\{ \left[\frac{(\boldsymbol{\theta}_i - \boldsymbol{\beta})^2}{2} - \phi_i(\boldsymbol{\theta}_k) \right] - \left[\frac{(\boldsymbol{\theta}_j - \boldsymbol{\beta})^2}{2} - \phi_j(\boldsymbol{\theta}_k) \right] \right\} \quad ,$$

with $D_{\Delta t}$ the *time delay distance* $\sim 1/H_0$ and model of the lens gives ϕ_i's.

Advantages of time delay measurements :

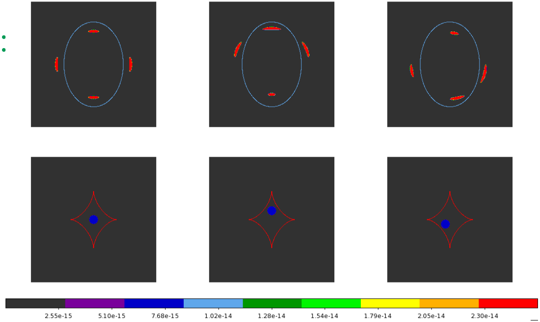
- direct measurement of distance ($\sim 5\%$), independent from local ladder,
- precision on H_0 is $\sim 7\%$, comparable with BAO $\Rightarrow$ complete other probes.

Example : RXJ1131-1231 of Suyu *et al.*2013.

Complications :

- sources can have a structure (e.g. Barnacka *et al.* 2014),
- lenses have structure too (e.g. Keeton, Moustakas 2009),
- there are structures on line-of-sight (need simulations),
- we know only a few "good" lenses (but Euclid will change that !).

Illustration :




```
||||| Cross |||||
Image 1 :
 NimagePoints = 3018
 Position = -1.65539e-23 -1.07854e-05
 Time Delay = 0.189156 years.
Image 2 :
 NimagePoints = 4138
 Position = -1.21018e-05 -1.47453e-21
 Time Delay = 0.238825 years.
Image 3 :
 NimagePoints = 3018
 Position = -7.21361e-22 1.07854e-05
 Time Delay = 0.189156 years.
Image 4 :
 NimagePoints = 4138
 Position = 1.21018e-05 -2.38803e-21
 Time Delay = 0.238925 years.
```

```
||||| Cusp |||||
Image 1 :
 NimagePoints = 4718
 Position = 1.12476e-05 4.90168e-06
 Time Delay = 0.232443 years.
Image 2 :
 NimagePoints = 4974
 Position = -1.71892e-21 1.17082e-05
 Time Delay = 0.189726 years.
Image 3 :
 NimagePoints = 2058
 Position = 4.60517e-22 -9.83243e-06
 Time Delay = 0.189301 years.
Image 4 :
 NimagePoints = 4718
 Position = -1.12476e-05 4.90168e-06
 Time Delay = 0.232443 years.
```

```
||||| Fold |||||
Image 1 :
 NimagePoints = 4232
 Position = 2.38267e-06 -1.09387e-05
 Time Delay = 0.192421 years.
Image 2 :
 NimagePoints = 2536
 Position = 1.45716e-06 1.01734e-05
 Time Delay = 0.190221 years.
Image 3 :
 NimagePoints = 3663
 Position = -1.24646e-05 -2.10585e-06
 Time Delay = 0.237939 years.
Image 4 :
 NimagePoints = 5441
 Position = 1.12099e-05 -3.062e-06
 Time Delay = 0.236583 years.
```

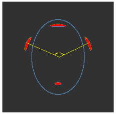
Radio-Flux Anomalies

Idea : given some images, we can fit their positions with an NSIE + γ_{ext}, but **in general we will not obtain the observed fluxes** of the different images.

Precisely : there is a **violation of universal magnification relations** :

- *cusp relation* :

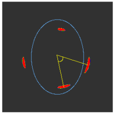
$$\lim_{\Delta\beta\to 0}\left[R_{\text{cusp}} \equiv \frac{\mu_1+\mu_2+\mu_3}{|\mu_1|+|\mu_2|+|\mu_3|}\right] = 0 \quad ,$$



with $\Delta\beta$ the offset between the source and the cusp of the caustic (μ_i are magnifications of cusp images).

- *fold relation* :

$$\lim_{\Delta\beta\to 0}\left[R_{\text{fold}} \equiv \frac{\mu_{\text{min}}+\mu_{\text{sad}}}{|\mu_{\text{min}}|+|\mu_{\text{sad}}|}\right] = 0 \quad ,$$



with $\Delta\beta$ the offset between the source and the fold caustic (μ_{min}, μ_{sad} resp. the magnif. of minimum – > 0 – and saddle – < 0 – images).

Past studies were studying the impact of **LoS structure** on these anomalies, nowadays people study more the **subhalos**. We want to use both !

References : Amara, Metcalf, *et al.* 2004 ; Xu *et al.* 2012 and 2014.

Adding structure along the LoS : κ vs Images

We observe a change in image positions (and corresponding critical lines).