About the curious appearance of half-integral post-Newtonian terms in the conservative dynamics of black-hole binaries

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Context: GW observatory at mHz frequency

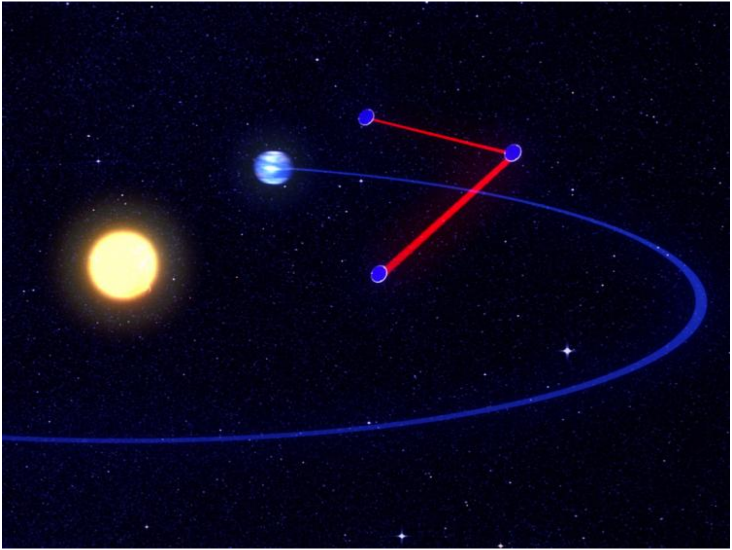
Gravitational-wave astronomy is emerging

- Advanced ground-based GW detectors soon ready to operate
  Frequency range: 10–10^4 Hz
- Pulsar-timing arrays searching for the radiation of supermassive black-hole binaries
  Frequency range: 10^{-9}–10^{-6} Hz
- Space-based observatory planed for 2034
  Frequency range: 10^{-3}–10^{-1} Hz (?)

"The L3 mission will study the gravitational Universe, searching for ripples in the very fabric of space–time created by celestial objects with very strong gravity, such as pairs of merging black holes."

ESA announcement (November 28th 2013)

eLISA: a possible proposal

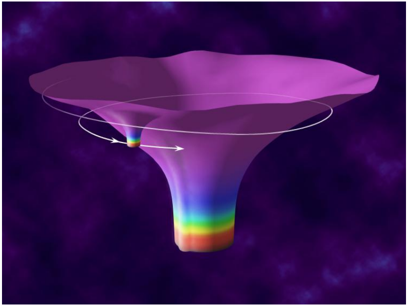


credit: AEI/MM/exozet

EMRI

Small compact object orbiting about a massive BH

$\hookrightarrow$ astrophysically relevant
can appear in galactic nuclei



credit: NASA

Potential source for eLISA

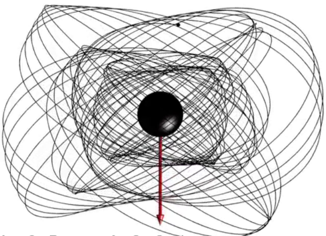
3 main formation scenarios:

1. Capture of compact stars
   - mass segregation
   - resonant relaxation
   - direct plunge or capture
2. Tidal disruption of binaries
3. Creation of massive stars in the accretion discs

Dynamics of EMRIs

- Eccentricity and orbit misalignment depend on the formation scenario
  - e very high in the capture scenario
  - e can be small for binary disruptions
  - in the first two scenario, the orbits are misaligned
- Significant strong field effects at periastron (for eccentric systems)
- Significant bursts of GW emission at periastron
  $\hookleftarrow$ important for the capture
- Complicated orbits with various timescales

114 days before merger, 36% of light speed



credit: S. Drasco & C. Cutler

Probe of the strong field region

$\Rightarrow$

TEST OF

- the nature of the central object
- GR in strong field regime

eLISA type mission:

expected rate	$1\mathrm{yr}^{-1}$	
mass range	m_1	$\sim 5M_\odot$–$20M_\odot$
	m_2	$\sim 10^5 M_\odot$–$10^6 M_\odot$
horizon	$z = 0.7$	
measured quantity	m_1	$\varepsilon = 0.1\%$
	m_2	$\varepsilon = 0.1\%$
	S_2	$\delta S_2 = 10^{-3}$
	e	$\delta e = 10^{-3}$

Specificity of EMRI waveform modeling

Accurate modeling crucial for

- detection
- parameter estimation

Failure of approaches normally used for black-hole binaries

- Numerical relativity
  $\hookrightarrow$ computational time issues
- Post-Newtonian approximation
  solution searched as a perturbative series in $(v_{\text{high. typ.}}/c)^2$

PN counting

$$1\text{PN} = 1/c^2$$

$\hookrightarrow$ converge badly for $q = m_1/m_2 \to 0$

Self-force approach

Covariant perturbative approach with $\varepsilon = q$

Self-force counting

SF $= q$ PSF$= q^2$

Method to obtain the SF equations of motion:

- Expand the perturbed metric to order ε

$$g^{(\varepsilon)}_{\mu\nu} \approx g_{\mu\nu} + \varepsilon\, \delta g^{(1)}_{\mu\nu}$$

- Describe the point-like objects with some $T^{\mu\nu} = \varepsilon\, T^{\mu\nu}_{(1)}[y^\alpha, g_{\alpha\beta}]$
- Solve the perturbed Einstein eqs with some "regular" Green function

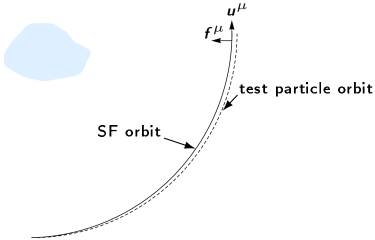
$$G^{\mu\nu}_{\text{reg}\;\alpha'\beta'}(x,x') = G^{\mu\nu}_{\text{R}\;\alpha'\beta'}(x,x') - G^{\mu\nu}_{\text{sing}\;\alpha'\beta'}(x,x')$$

highly non-trivial to construct with full rigor

Self-force equations of motion

- equations of motion = geodesic equations in $g_{\mu\nu} + \varepsilon\, \delta g^{(1)\,\mathrm{reg}}_{\mu\nu}$

$$\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = f^\mu = \mathcal{O}(q)$$



- (Delicate) numerical integration required in general
- Use of $Y^{\ell m}$-mode-sum regularization in practice

$$f^{\mathrm{reg}} = \sum_{\ell=0}^{+\infty} \left[f_\ell - A_\pm L - B - B - C L^{-1} \right] - D \qquad \text{with } L = \ell + 1/2$$

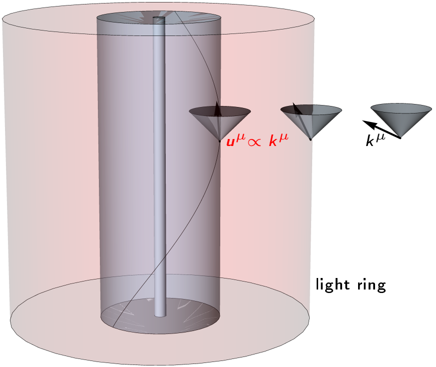
Case of exact circular motion

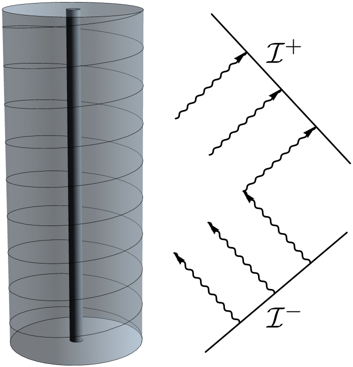
System of interest

small mass particle in circular orbit about a black hole

Consequences:

- Helical killing vector: $k^\mu \to \partial_t^\mu + \Omega\, \partial_\phi^\mu$ in some class of gauges
- Presence of incoming waves $\Rightarrow$ conservative system





Detweiler redshift observable

Gravitational redshift

$$z = -k^\mu u_\mu$$

Properties:

- Gauge invariance property (for helically-symmetric gauge vectors)

  z is a physical observable

- Simple interpretation for far-away observers along the axis
- $u^\mu = z^{-1} k^\mu$ and $z^{-1} = u^0$

$\Rightarrow$ u^0 can be used for comparison with PN results

Comparison with PN calculations

General motivations:

- PN and SF perturbative formalisms both fairly delicate
- Both methods involve non-trivial regularizations
  - For PN:
    - effective representation of bodies by point-particle $\Rightarrow$ dim-reg
    - technical Finite Part regularization to treat the field multipole expansion
  - For SF:
    - Green function regularization
    - mode-sum regularization for $\delta g^{(1)}_{\mu\nu}$ (or other equivalent reg)
- Insight provided by each formalism on the other about:
  - physical content
  - convergence properties

Recent SF computations of u^0 PN expansion to very high orders

$$u^0 = \left[-\left(g_{\mu\nu} + \delta g^{\text{reg}}_{\mu\nu}\right) \frac{v^\mu v^\nu}{c^2} \right]^{-1/2}$$

Work of Shah, Friedman & Whiting: 10.5PN order

- Start from Teukolsky radial equation for $\Psi_0 = C_{\mu\alpha\nu\beta} l^\mu m^\alpha l^\nu m^\beta$:

$$\mathcal{L}^{(r)}_{\text{Teuk}}({}_2R_{\ell m\omega})[\delta g] = \tau^{(r)}_{\ell m\omega} S[\delta g]$$

$\hookrightarrow$ solutions built from the hom sol of Mano, Suzuki & Tagasuki

appearance of series of hypergeometric functions

- Accurate numerical computation with 350 digits!!
- Extract the metric angular modes in some "radiation gauge"
- Evaluate the field at particle position by mode-sum regularization
- Calculation of u^0; reconstruction of some analytic coefficients

High PN orders from SF calculations II

Work of Bini and Damour: 8.5PN order

- Start from the Regge-Wheeler radial equations for the master function:

$$\mathcal{L}_{\mathrm{RG}}^{(r)} R_{\ell m\omega}^{(\mathrm{odd})} = S_{\ell m\omega}^{(\mathrm{odd})}(r)$$

$\hookrightarrow$ solutions at particule position built from Mano, Suzuki & Tagasuki

- Computation for $\ell \leq 4$ using the hypergeometric series (for "up" sol)
- PN resolution for arbitrary ℓ
- Results "patched" together motivated by phys/math arguments

$\Rightarrow$ fully analytic expressions

Half-integral PN order terms in u^0

Definition

$$u^0 = \frac{1}{\sqrt{1-3y}} + q\, u^0_{\rm SF} + \mathcal{O}(q^2)$$

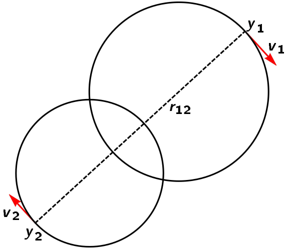
PN parameter: $y = \left(\frac{G m_2 \Omega}{c^3}\right)^{2/3}$

$$u^0_{\rm SF} = -y + \sum_{n=-2}^{10} \alpha_n y^{n+1} + \sum_{n=4}^{10} \beta_n y^{n+1} \ln y + \sum_{n=7}^{10} \gamma_n y^{n+1} \ln^2 y$$
$$- \frac{13696}{525}\pi y^{13/2} + \frac{81077}{3675}\pi y^{15/2} + \frac{82561159}{467775}\pi y^{17/2}$$
$$+ \text{higher half-integral orders} \ + y\, \mathcal{O}(y^{11})$$

Notice the presence of terms $\propto 1/c^{2q+1}$

- Why do they appear for this conservative dynamics?
- Can we compute them using standard PN techniques?

Is there a paradox?



$$\left(u^0\right)_{\text{inst}} \sim \sum_{j,k,p,q} \nu^j \left(\frac{Gm}{r_{12}c^2}\right)^k \left(\frac{\boldsymbol{v}_{12}^2}{c^2}\right)^p \left(\frac{\boldsymbol{n}_{12}\cdot\boldsymbol{v}_{12}}{c}\right)^q$$

(ν: $\frac{m_1 m_2}{m^2}$; $\boldsymbol{v}_{12}$: $\boldsymbol{v}_1 - \boldsymbol{v}_2$)

$$n = 2k + 2p + q + 2$$

For circular orbits $\boldsymbol{n}_{12}\cdot\boldsymbol{v}_{12} = 0$

What if they were non-"instantaneous" terms? $\boldsymbol{n}_{12}(t)\cdot\boldsymbol{v}_{12}(t') \neq 0$

Well-known hereditary effect in classic field theory

the tail effect

HUYGENS PRINCIPLE

"Il y a encore à considérer dans l'émanation de ces ondes, que chaque particule de la matière, dans laquelle une onde s'étend, ne doit pas communiquer son mouvement seulement à la particule prochaine, qui est dans la ligne droite tirée du point lumineux, mais qu'elle en donne aussi nécessairement à toutes les autres qui la touchent et qui s'opposent à son mouvement."

C. Huygens, Traité de la lumière

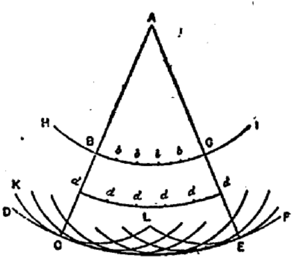


Fig. 6.

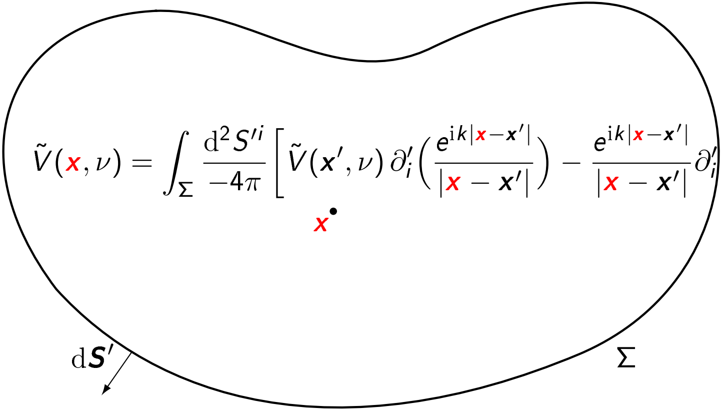
credit: Caspar Netscher (ca 1639–1684)

Validity of Huygens principle

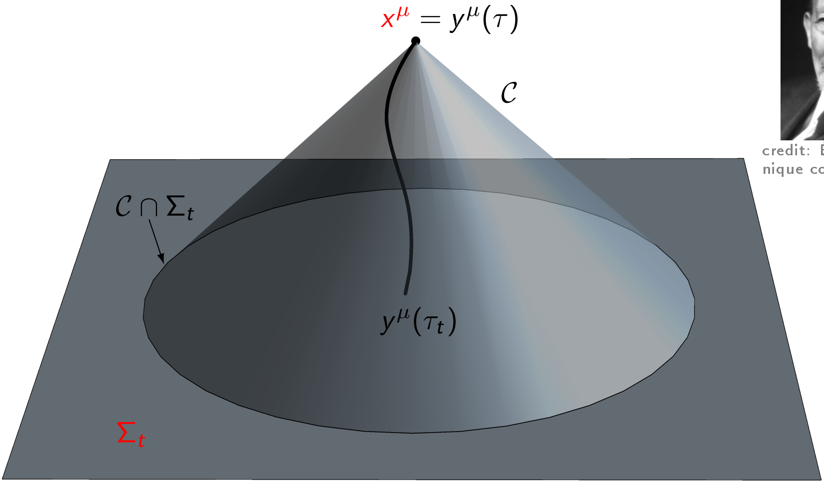
V solution of: $\Box V = 0$

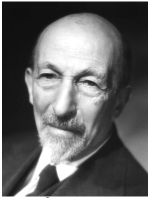
$$V(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \mathrm{d}\nu \, \tilde{V}(\mathbf{x}, \nu) e^{-2\pi i \nu t}$$

$$\tilde{V}(\mathbf{x}, \nu) = \int_{\Sigma} \frac{\mathrm{d}^2 S'^i}{-4\pi} \left[\tilde{V}(\mathbf{x}', \nu) \, \partial'_i \left(\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \right) - \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \partial'_i \tilde{V}(\mathbf{x}', \nu) \right]$$



Past dependence of a wave-type field $\Phi(x^\mu)$ on $\mathcal{C} \cap \Sigma_t$

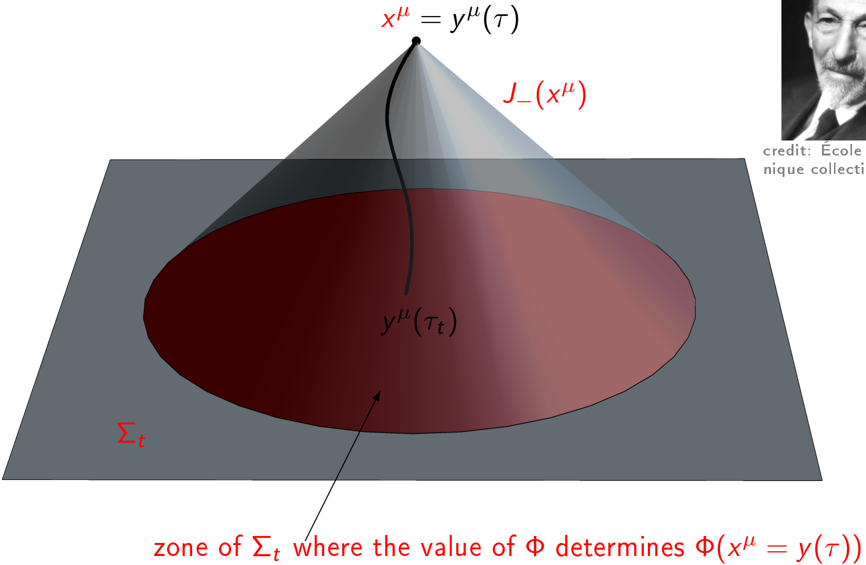


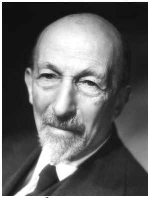


credit: École polytechnique collections

Tail effects

Past dependence of a wave-type field $\Phi(x^\mu)$ inside $\mathcal{C}$





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Green function in $(d+1)$ dimensions I

Sourced wave equation: $\Box\Phi = \rho$

Retarded solution: $\Phi(x) = \int \mathrm{d}^4 x' G_\mathrm{R}(x, x')\rho(x')$

General formula for the retarded Green function $G_\mathrm{R}(x, x') \equiv G(x - x')$

$$G(x) = -\theta(t)\int_0^{+\infty} \mathrm{d}k \left(\frac{k}{r}\right)^{\frac{d-2}{2}} \sin(ckt) J_{\frac{d-2}{2}}(kr)$$

Green function in $(d+1)$ dimensions II

- Reduction for d odd:

$$G(x) = \frac{(-1)^{\frac{d-1}{2}}}{4\pi^{\frac{d-1}{2}}} \left(\frac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{d-3}{2}} \frac{\delta(x^0 - r)}{r}$$

- Reduction for d even:

$$G(x) = \frac{(-1)^{\frac{d}{2}}}{2\pi^{\frac{d}{2}}} \left(\frac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{(d-2)}{2}} \int_r^{+\infty} \mathrm{d}s \, \frac{\delta(x^0 - s)}{\sqrt{s^2 - r^2}}$$

Green function in curved spacetime

Focus on the equation satisfied by $\gamma_{\mu\nu} = \delta g_{\mu\nu} - \frac{1}{2}(g^{\alpha\beta}\delta g_{\alpha\beta})g_{\mu\nu}$

Sourced wave equation: $\Box\gamma^{\mu\nu} + 2R_{\alpha\ \beta}^{\ \mu\ \nu}\gamma^{\alpha\beta} = \frac{16\pi G}{c^4}T^{\mu\nu}$

Retarded solution: $\gamma^{\mu\nu}(x) = \frac{4G}{c^4}\int \mathrm{d}^4x'\sqrt{-g'}\,G_{\mathrm{R}\ \alpha'\beta'}^{\mu\nu}(x,x')\,T^{\alpha'\beta'}(x')$

Hadamard form for the retarded Green function

$$G_{\mathrm{R}\ \alpha'\beta'}^{\mu\nu}(x,x') = \theta_+(x,\Sigma)\left[U^{\mu\nu}{}_{\alpha'\beta'}\delta(\sigma) + V^{\mu\nu}{}_{\alpha'\beta'}\theta(-\sigma)\right]$$

Perturbative interpretation

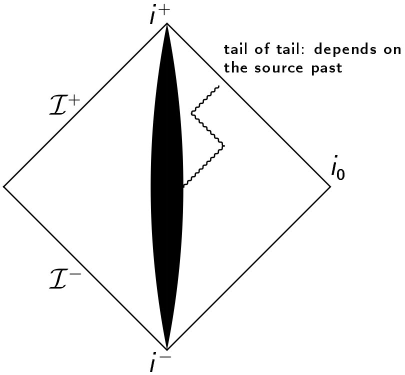
Tail wave

GW scattered on the curvature of space-time

$\hookrightarrow$ corresponds to a $M \times M_L$ interaction in the waveform

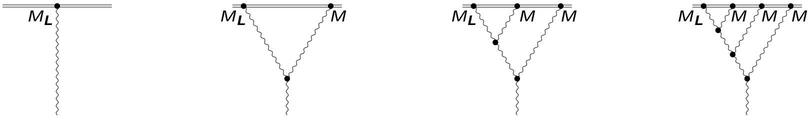
Tail-of-tail wave

tail wave scattered on the curvature of space-time



$\hookrightarrow$ associated with the fact that null geodesic $\neq$ 'straight lines'

EFT point of view:



Post-Newtonian approximation

Small post-Newtonian parameter:

$$v/c \ll 1$$

highest characteristic velocity

For a PN field Q: $\partial_t Q/c \sim \varepsilon\,\partial_i Q$

post-Newtonian expansion:

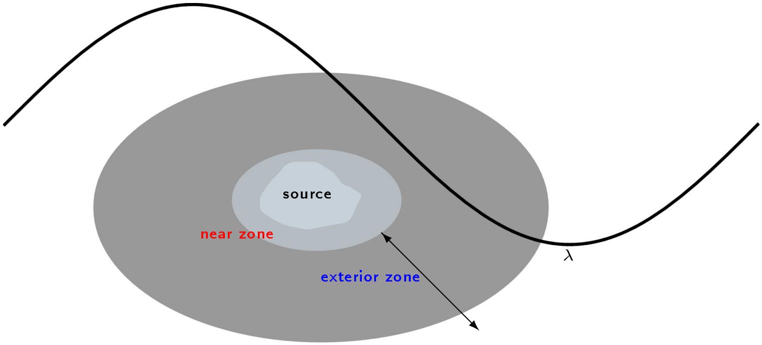
expansion in the small parameter $\varepsilon = v/c$

For ordinary matter: $T^{00} \sim c^2 \qquad T^{0i} \sim c \qquad T^{ij} \sim c^0$

Isolated PN sources

Physical assumptions:

- Ordinary matter source $T^{\mu\nu}$ with compact support
- No-incoming wave condition
- Post-Newtonian approximation $v/c \ll 1$ valid within the matter $\Rightarrow D_{\text{matt}} \ll \lambda$



Multipolar post-Minkowskian approximation

Small post-Minkowskian parameter:

$$Gm/(Lc^2) \ll 1$$

m typical mass — L typical "size"

… but for a propagating field Q: $\partial_t Q/c \sim \partial_i Q$

Multipolar expansion:

expansion in the small parameter $(D_{\mathrm{matt}}/\lambda)$

$$\Rightarrow$$

expansion in $1/r$ at $t - r/c = $cst

For Φ satisfying $\Box\Phi = \rho$

$$\Phi_{\mathrm{ext}} = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left(\frac{M_L(t - r/c)}{r} \right)$$

Einstein equations in harmonic gauge

PM perturbation: $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$

Harmonic gauge condition

$$\nabla^\nu \nabla_\nu x^\mu = 0 \quad \Leftrightarrow \quad \partial_\nu h^{\mu\nu} = 0$$

Relaxed Einstein equations

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu} \equiv \frac{16\pi G}{c^4} |g| T^{\mu\nu} + \Lambda^{\mu\nu}(\partial h, \partial h)$$

- $\Lambda^{\mu\nu}$ contain a 2[nd] order derivative: $-h^{\alpha\beta}\partial_{\alpha\beta} h^{\mu\nu}$
  $\hookrightarrow$ produces the non-linear tail effect
- the gauge condition implies the equations of motion

Standard PN scheme

Smooth $T^{\mu\nu}$ assumed with

$T^{00} \sim c^2 \qquad T^{0i} \sim c \qquad T^{ij} \sim c^0$

Iterative computation of $h^{\mu\nu}_{[m]}$

- Assume $h^{\mu\nu}_{[m']}$ known for $m' < m$ (not needed at leading order)
- Solution for $h^{\mu\nu}_{[m]}$ taken to be

$$h^{\mu\nu}_{[m]} = \frac{16\pi G}{c^4}\left\{\Box_{\mathrm{R}}^{-1}\left[\tau^{\mu\nu}(h^{\alpha\beta})\right]\right\}_{[m-4]}$$

- Go to the next order

Formal PN series:

$$\overline{h^{\mu\nu}} = \sum_{m=m_{\min}}^{+\infty} \frac{1}{c^m} h^{\mu\nu}_{(m)}$$

Main issue of PN expansion

What is the iterative PN expansion $\overline{\Box_{\mathrm{R}}^{-1}\tau}$ of
$$\Box_{\mathrm{R}}^{-1}\tau = \int \frac{d^3\mathbf{x}'}{-4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tau(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c) \text{ ?}$$

$$\overline{\Box_{\mathrm{R}}^{-1}[\tau]} = \sum_{k\geq 0, n} \frac{(-1)^k}{k!} \frac{\partial_t^k}{c^k} \int \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}-\mathbf{x}'|^{k-1} \frac{\overline{\tau}_{(n)}(\mathbf{x}', t)}{c^n}$$

does not make sense!!

information on the field behavior far from the system required...

Multipolar PM expansion in the vacuum

Key idea to study the field outside the near zone

$h^{\mu\nu}$ in the exterior zone satisfies the vacuum Einstein equations
$\hookrightarrow$ contained in the most general PM asymptotic solution

Principle of the algorithm:

- Decompose $h^{\mu\nu}$ as $\displaystyle\sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}$
- Find iteratively the most general solution of

$$\Box h^{\mu\nu}_{(n+1)} = \Lambda^{\mu\nu}_{(n)}(\partial h_{(\leq n)}, \partial h_{(\leq n)})$$

- Absorb homogeneous solutions by moment redefinitions

Finite Part (FP) regularization

PM solutions expressed in terms of FP integrals

$\mathrm{FP}\int \mathrm{d}^3\mathbf{x}' F(\mathbf{x}',t)$ for a smooth function F on $\mathbb{R}^{*3}$:

- Computation of $I[F](B) \equiv \int \mathrm{d}^3\mathbf{x}' (|\mathbf{x}'|/r_0)^B F(\mathbf{x}')$
- Expansion of $I[F](B)$ in a Laurent series of the form

$$\sum_{k=k_0}^{+\infty} I_k[F] B^k$$

- $\mathrm{FP}\int \mathrm{d}^3\mathbf{x}' F(\mathbf{x}',t) = I_0[F]$ $\leftarrow$ depends on r_0

Linearized exterior field I

Linearized Einstein equations in vacuum:

$$\Box h^{\mu\nu}_{(1)} = 0 \qquad \partial_\nu h^{\mu\nu}_{(1)} = 0$$

with the no-incoming wave condition

$$\lim_{\substack{r\to+\infty \\ t+r/c\to\text{cst}}} h^{\mu\nu}_{(m)} = 0 \qquad \lim_{\substack{r\to+\infty \\ t+r/c\to\text{cst}}} \left[\left(\partial_r + \frac{1}{c}\partial_t\right)\left(r\, h^{\mu\nu}_{(m)}\right)\right] = 0$$

distance to the origin $|x|$

Form of the most general solutions in Minkowskian-like coordinates:

- in spherical symmetry $\dfrac{I(t-r/c)}{r}$
- in general $\displaystyle\sum_{\ell\geq 0} \partial_{IL}\left(\frac{I_{JL}(t-r/c)}{r}\right)$

  (possible contraction to ε_{abc} for current moments)

Linearized exterior field II

Most general exterior linear solution

$$h^{\mu\nu}_{(1)} = h^{\mu\nu}_{\text{can}}(I_L(t'), J_L(t')) + \text{linear gauge transformation term in } \phi^{\mu}_{(1)}$$
$$\text{with } \Box\phi^{\mu}_{(1)} = 0$$

$\Rightarrow h^{\mu\nu}$ entirely parameterized by 6 moments

- I_L= source mass-type moment of order ℓ
  J_L= source current-type moment of order ℓ
- 4 gauge moments $\Leftrightarrow$ high-order PN corrections to $\{I_L, J_L\}$

Unicity of the multipole parameterization iff the moments are STF

e.g. $I_{ij} = I_{ji},\ I_{ii} = 0$

Post-Minkowskian iteration

- Search of a particular solution of $\Box h^{\mu\nu}_{(n+1)} = \Lambda^{\mu\nu}_{(n+1)}(h_{(\leq n)}, h_{(\leq n)})$
  $\Box^{-1}_{\mathrm{R}} \Lambda^{\mu\nu}_{(n+1)}$ ill-defined... but

$$\Box(\mathrm{FP}\Box^{-1}_{\mathrm{R}} F) = F$$

  solution under assumption of past stationarity : $p^{\mu\nu}_{(n+1)} = \mathrm{FP}\Box^{-1}_{\mathrm{R}}\Lambda^{\mu\nu}_{(n+1)}$

- Determination of the homogeneous solution $q^{\mu\nu}_{(n+1)}$ of $\Box h^{\mu\nu} = \dots$

$$\partial_\nu h^{\mu\nu}_{(n+1)} = \partial_\nu p^{\mu\nu}_{(n+1)} + \partial_\nu q^{\mu\nu}_{(n+1)} = 0 \quad \text{and} \quad \Box q^{\mu\nu}_{(n+1)} = 0 \quad \Rightarrow \quad q^{\mu\nu}_{(n+1)}$$

General solution

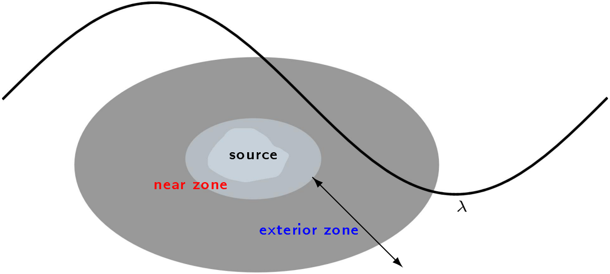
$$h^{\mu\nu}_{(n+1)} = p^{\mu\nu}_{(n+1)} + q^{\mu\nu}_{(n+1)}$$

(+ homogeneous solution absorbed in a moment redefinition)

Principle of the matching procedure

- In the exterior zone D_{ext}: $h^{\mu\nu} = \mathcal{M}(h^{\mu\nu}) \equiv$ mult expansion of $h^{\mu\nu}$
- In the near zone D_{near}: $h^{\mu\nu}$ given by the searched PN expression $\overline{h}^{\mu\nu}$
- In some buffer zone $D_{\mathrm{near}} \cap D_{\mathrm{ext}}$
- stationarity in the remote past

$$\overline{h}^{\mu\nu} = \mathcal{M}(h^{\mu\nu})$$



Expansion of retarded quantities

Result of matching

$$\overline{\Box_{\mathrm{R}}^{-1}}[\tau] = \overline{\Box_{\mathrm{R}}^{-1}}[\tau] + \mathcal{H}[\tau]$$

with $$\overline{\Box_{\mathrm{R}}^{-1}}[\tau] = \sum_{k\geq 0,n} \frac{(-1)^k}{k!} \frac{\partial_t^k}{c^k} \mathrm{FP} \int \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}-\mathbf{x}'|^{k-1} \frac{\overline{\tau}_{(n)}(\mathbf{x}',t)}{c^n}$$

$$\mathcal{H}[\tau] = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!} \hat{\partial}_L \left\{ \frac{\mathcal{R}[\tau]_L(t-r/c) - \mathcal{R}_L[\tau](t+r/c)}{2r} \right\}\Bigg]$$

- $\mathcal{H}[\tau]$ is a homogeneous solution of the wave equation
  $\hookrightarrow$ the source "feels" some external-like regular wave

$\mathcal{H}[\tau]$ actually contains the tail effect

- $\mathcal{R}_L$ depends on $\mathcal{M}(h)$

Structure of tail terms

Interactions involving $k_{\text{non stat}} \geq 2$ non-static moments ignored

$M \times \ldots \times M \times I_P$ tail interactions in $h^{\mu\nu}$

$$h^{\alpha\beta}_{M\times\cdots\times M\times I_P} \sim \sum_{k,p,\ell,i} \frac{G^k M^{k-1}}{c^{3k+p}} \hat{n}^L \left(\frac{r}{c}\right)^{\ell+2i} \int_{-\infty}^{+\infty} \mathrm{d}u\, \kappa^{\alpha\beta}_{LP}(t,u)\, I^{(a)}_P(u)$$

$n^L = n^{i_1} \ldots n^{i_\ell}$; # of time der

- a= # of ∂_t $= k + p + \ell + 2i + 1$
- power of $1/c$: $n = 3k + p + \ell + 2i + s - 2$
- angular momentum selection rules:
  - $|\ell - p| \leq s$ [with $s(h^{00}) = 0$, $s(h^{0i}) = 1$, $s(h^{ij}) = 2$]
  - s and $\ell - p$ have same parity

$$\Rightarrow n = 3k + 2p + 2j - 2 \text{ with } j \in \mathbb{N}$$

What are the first half-integral PN orders?

Half-integral PN orders below 7.5PN

$$n = 11 + 2(p-2) + 2j$$

Only $k = 3$ contributes: $M \times M \times \text{moment} = $ tail of tail

- 5.5PN, 6.5PN, 7.5PN, ... for the mass quadrupole
- 6.5PN, 7.5PN, ... for the mass octupole
- 7.5PN, ... for the mass hexadecapole

Orders for current-type interaction: involve $\varepsilon_{ija} J_{aL-1}$
$\hookrightarrow$ deduced from $a \to a-1$, $p \to p+1$

- 6.5PN, 7.5PN, ... for the current quadrupole
- 7.5PN, ... for the current octupole

Construction of the cubic field

Strategy

Compute the relevant part of $\mathcal{H}[\tau]$ through that of $\overline{\mathcal{M}(h^{\mu\nu})}$

- Start from the MPM quadratic field $M \times$moment
- Construct the cubic source $M \times M \times$moment

$$N_{(3)}^{M \times M \times M_L} = \sum_{\ell}^{\text{finite}} S_\ell(r, t - r/c)\hat{n}^L$$

$\hookrightarrow$ contains
  - "instantaneous" terms: $S(r, t - r/c) = r^{B-k} F(t - r/c)$
  - hereditary terms (related to tails):

$$S(r, t-r/c) = r^{B-k} \int_1^{+\infty} \mathrm{d}x\, Q_m(x)\, F(t - r\, x/c)$$

- Apply $\Box_R^{-1}$ on each source piece $S(r, t - r/c)\hat{n}^L$

Extraction of the hereditary tail part

- Variant of the previous formula for $\overline{\Box_R^{-1}[\ldots]}$ for $r \to 0$

$$h = \underbrace{\overline{\hat{\partial}_L\left(\frac{G(t-r/c)-G(t+r/c)}{r}\right)}}_{\text{contains all hereditary contributions}} + \underbrace{\Box^{-1}_{\text{inst}}[\overline{S}\,\hat{n}^L]}_{\text{local-in-time operator}}$$

- Analysis of $G(u) = C_{k,\ell,m} \times \mathrm{FP}_{B=0}$ (some integral with a kernel τ^B)

$$\Rightarrow \quad G_{\text{tail-tail}}(u) \propto \frac{G^3M^2}{c^n}\text{Res } C_{k,\ell,m}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau\,M_L^{(a)}(u-\tau)$$

- Ansatz: the conservative part is given by

$$G_{\text{cons}}(u) \propto \frac{G^3M^2}{c^n}\text{Res } C_{k,\ell,m}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau\left(\frac{M_L^{(a)}(u-\tau)+M_L^{(a)}(u+\tau)}{2}\right)$$

PN ITERATION

By matching
$$h^{00} = h^{00}_{\text{naive}} + h^{00}_{\text{tail}}$$
$$= -4\frac{U}{c^2} + \ldots + h^{00}_{\text{LO tail}} + \ldots$$

Likewise for h^{0i}, h^{ij}

$\Rightarrow$ Insertion into $\Lambda_{(2)}$ and $\Lambda_{(3)}$ can generate coupling of U, ..., and $h^{\mu\nu}$

- Gauge transformation to minimize the couplings: terms $0i$, $ij \to 00$
- EE must be iterated with coupling included
  - 2 iterations needed for I_{ij}
  - 1 iteration needed for I_{ijk} and J_{ij}
- Equations to be solved of the form

$$\Delta\Psi_L = \hat{x}^L r^{2p} \phi$$

$\longleftrightarrow$ systematic use of superpotentials: $\Delta\phi_{2k+2} = \phi_{2k}$

From the metric to u^0

Structure of the $\frac{n}{2}$PN type grav field

$$h^{\mu\nu} \sim \sum G^{(a)}_{\rm cons}(t)\,\hat{x}^L \partial\phi$$

- Superpotentials obtained by guess work $\rightarrow$ to be regularized at $\boldsymbol{x} = \boldsymbol{y}_1$
- I_{ij}, I_{ijk}, ... replaced by their explicit PN values
- Integrals computed by using

$$x^i_{12}(t \pm \tau) = \cos(\Omega\tau)\, x^i_{12}(t) \pm \sin(\Omega\tau)\, v^i_{12}(t)/\Omega\,,$$
$$v^i_{12}(t \pm \tau) = \mp\Omega\, \sin(\Omega\tau)\, x^i_{12}(t) + \cos(\Omega\tau)\, v^i_{12}(t)$$

and

$$\int_0^{+\infty} \mathrm{d}\tau \ln\tau\, e^{\mathrm{i}\lambda\tau} = -\frac{\pi}{2|\lambda|} - \frac{\mathrm{i}}{\lambda}(\ln|\lambda| + \gamma_E)$$

Note that for $\lambda = m\Omega$ the term $\propto \pi$ is invariant for $t \rightarrow -t$

Conclusion

- We find that there is no radial velocity
- We confirmed the origin of the half-integral order conservative terms
- We found full agreement with the SF results
- Generalization beyond linear order in m_1/m_2 could be interesting

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Blanchet, F. & Whiting Phys. Rev. D 90, 044017 (2014)

and references there in