Motion of small bodies in curved spacetime: An introduction to gravitational self-force

Adam Pound

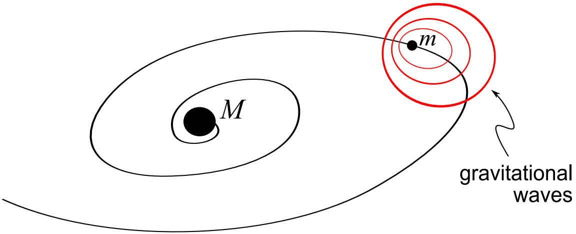
University of Southampton

18 November 2013

Motivation: source for eLISA

Extreme-mass-ratio inspiral (EMRI)

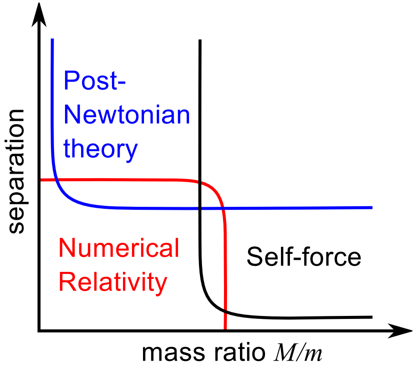
- $\sim$1–10$M_{\odot}$ neutron star or black hole orbits supermassive black hole
- m emits gravitational radiation, loses energy, spirals into M
- $\sim$1–100 events detectable in eLISA's lifetime
- waveforms carry information about strong-field dynamics and structure of spacetime near black hole
- need to model motion of small body



More motivation

Interface between models

- establish benchmarks for $m \ll M$ limit of post-Newtonian theory and numerical relativity
- fix high-order post-Newtonian parameters
- fix Effective One Body parameters



Modeling IMRIs and similar-mass binaries

- self-force has surprisingly large domain of validity [Le Tiec et al]
- potentially accurate model of intermediate-mass-ratio and even similar-mass binaries

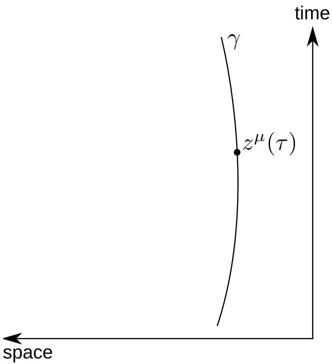
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore h_{\mu\nu}$ depends on past history



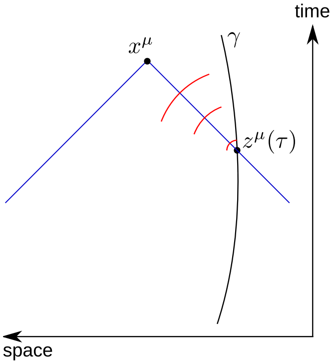
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



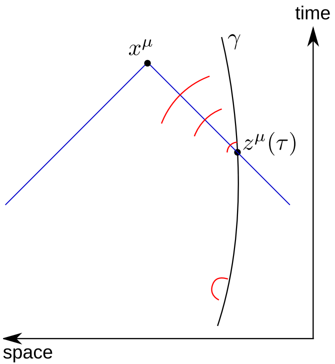
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



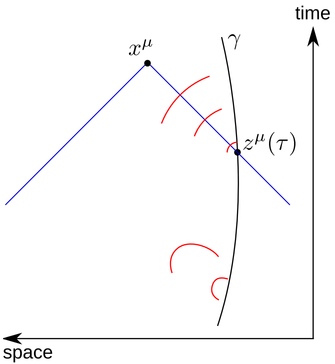
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



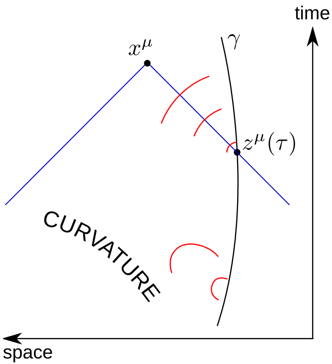
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



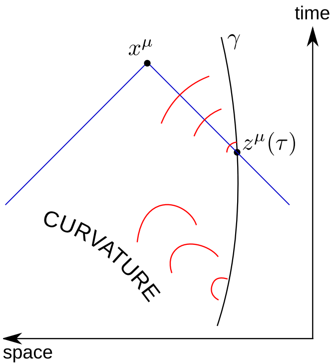
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore h_{\mu\nu}$ depends on past history



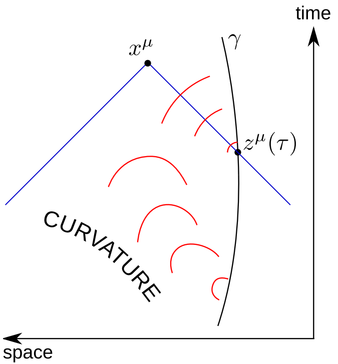
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



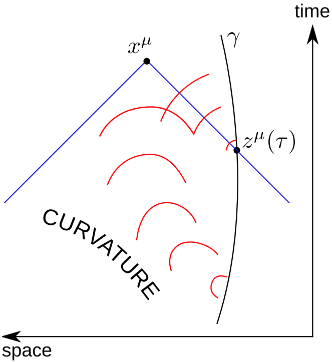
Point particle picture

Linearized theory

- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



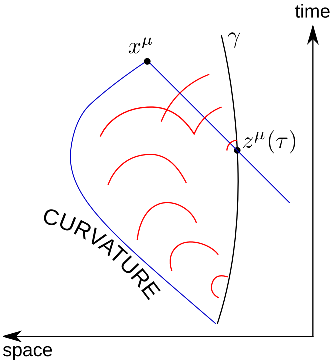
Point particle picture

Linearized theory

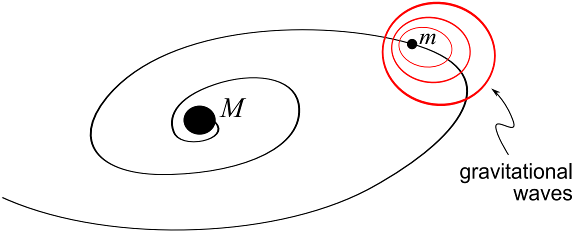
- treat m as point particle, with stress-energy $T^{\mu\nu} \sim m\delta^3(x^\rho - z^\rho)$
- write total metric as $g_{\mu\nu} + h_{\mu\nu}$ (e.g., $g_{\mu\nu}$ is metric of M, $h_{\mu\nu}$ is created by m)
- approximate Einstein equation $G_{\mu\nu}[g+h] = 8\pi T_{\mu\nu}$ with *linearized EFE* $\delta G^{\mu\nu}[h] = 8\pi T^{\mu\nu}$

Tail

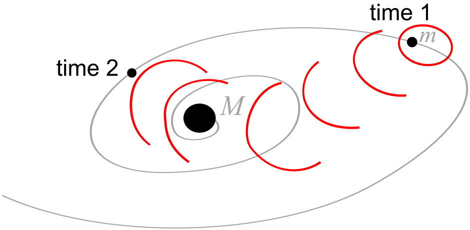
- part of perturbation propagates slower than light
- light "cone" bends $\therefore$ $h_{\mu\nu}$ depends on past history



Extreme-mass-ratio inspirals



Extreme-mass-ratio inspirals



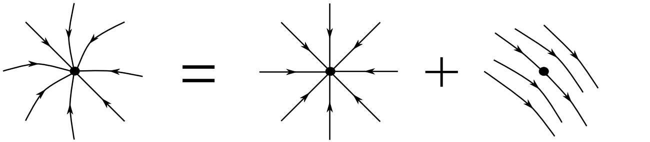
Self-force: geodesic motion in an effective metric

MiSaTaQuWa equation [Mino,Sasaki,Tanaka, & Quinn,Wald]

- nonlocal tail acts as potential, exerts force $F^\mu \sim m\nabla^\mu\text{tail}$
- tail isn't nice: non-differentiable, not a solution to a field equation

Generalized equivalence principle [Detweiler-Whiting]

- local field near particle split into two: $h^{(1)}_{\mu\nu} = h^{\mathrm{S}(1)}_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu}$
- $h^{\mathrm{S}(1)}_{\mu\nu} \sim \frac{m}{r} + O(r^0)$; local bound field of particle
- $h^{\mathrm{R}(1)}_{\mu\nu} \sim \text{tail} + \text{local terms}$; smooth solution to source-free EFE
- motion is geodesic in effective metric $g_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu}$



body's field $h_{\alpha\beta}$ = singular field $h^S_{\alpha\beta}$ + regular field $h^R_{\alpha\beta}$

Outline

Outline

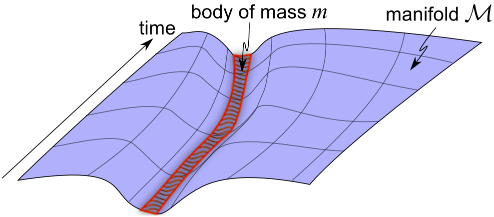
A small extended body moving through spacetime

Fundamental question

- how does a body's gravitational field affect its own motion?

Regime: small body

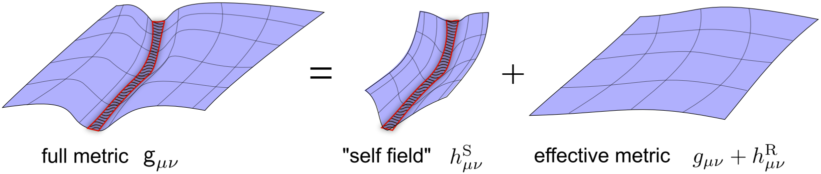
- examine spacetime $(\mathcal{M}, g_{\mu\nu})$ containing body of mass m and external lengthscales $\mathcal{R}$
- seek representation of body's motion when its mass and size are $\ll \mathcal{R}$



Non-perturbative approach [Harte 2011]

Non-perturbative decomposition

- split metric into "self-field" generated by body and slowly varying remainder



Equation of motion

- body moves as test body in effective metric $g_{\mu\nu} + h^{R}_{\mu\nu}$: motion is geodesic except for coupling of body's multipole moments $I \sim \int_{body} T^{\mu\nu}$ to curvature of effective metric

However...

Material body

- integrals over body's interior preclude description of black hole

Field

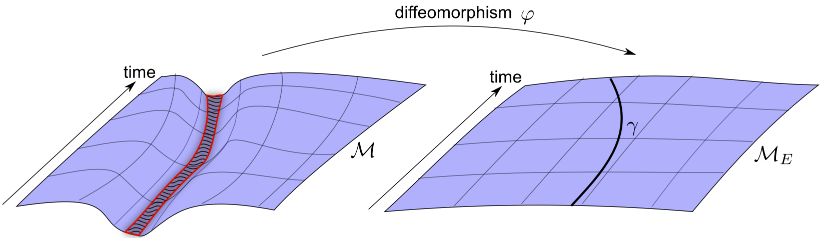
- describing motion in terms of metric isn't sufficient: we need a means of solving the EFE to obtain the metric (and a means of isolating the effective metric)
- minor drawback: Harte's effective metric doesn't satisfy vacuum EFE $\Rightarrow$ not a "nice" generalization of Detweiler-Whiting field

Perturbation theory

- treat body as source of perturbation of external background spacetime $(\mathcal{M}_E, g_{\mu\nu})$:

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \dots$$

- ϵ counts powers of m
- assume body is compact, so as $m \to 0$, linear size $\to 0$ at same rate
- seek representation of motion in $(\mathcal{M}_E, g_{\mu\nu})$



Body in exact spacetime

Representation of motion in external spacetime

Challenges in applying perturbation theory

Multiple time scales in binary

- orbital period ($\sim M$)
- time over which inspiral occurs ($\sim M^2/m$)

Multiple length scales

- near small object: scale of object's size ($\sim m$)
- everywhere else: scale of external universe ($\sim M$)

Identifying object's position, spin, higher moments

- point particle not valid in nonlinear field theory such as GR
- how do we capture bulk parameters without worrying about details of object's composition?
- how do we best represent the small object's bulk motion (e.g., identify its "center")?

Approach I [Gralla & Wald 2008]: power series

Expansion of EFE

- expand metric in Taylor series:

$$\mathsf{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x) + \epsilon^2 h^{(2)}_{\mu\nu}(x) + \ldots$$

- solve EFE order by order *outside body*:

$$\begin{aligned} \delta G_{\mu\nu}[h^{(1)}] &= 0 \\ \delta G_{\mu\nu}[h^{(2)}] &= -\delta^2 G_{\mu\nu}[h^{(1)}] \\ &\vdots \end{aligned}$$

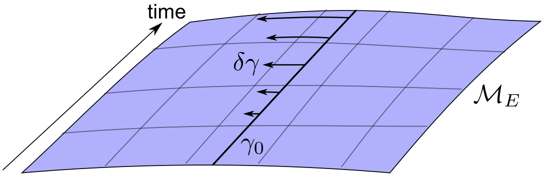
- motion determined by Bianchi identity

Representation of motion in power series

$$z^\mu(\tau,\epsilon) = z^\mu_{(0)}(\tau) + \epsilon z^\mu_{(1)}(\tau) + \epsilon^2 z^\mu_{(2)}(\tau) + \ldots$$

Meaning

- $z^\mu_{(0)}$ identified as remnant of body at $\epsilon = 0$
- self-force corrections accounted for by deviation vectors $z^\mu_{(n)}(\tau)$



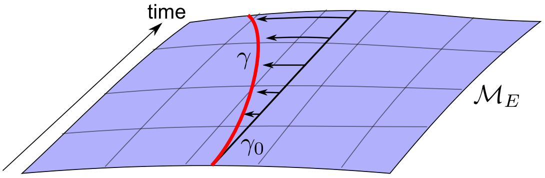
Problem

- as body drifts away from γ_0, corrections $z^\mu_{(n)}(\tau)$ grow large
- representation of motion only meaningful and accurate for short time

Approach II [Pound 2010]: self-consistent expansion

Unexpanded worldline

- rather than finding deviation vectors $z^{\mu}_{(n)}$, seek a worldline $z^{\mu}(\tau, \epsilon)$ that faithfully tracks body's bulk motion



Self-consistent expansion

- since $h_{\mu\nu}$ depends on γ, can't expand $h_{\mu\nu}$ in regular power series without also expanding γ
- allow γ to depend on ϵ and assume expansion of form

$$\mathrm{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + h_{\mu\nu}(x;\gamma)$$
$$= g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x;\gamma) + \epsilon^2 h^{(2)}_{\mu\nu}(x;\gamma) + \ldots$$

Finding the field in a self-consistent expansion

Expansion of EFE

- need a method of systematically solving for each $h^{(n)}_{\mu\nu}$
  $\Rightarrow$ use method analogous to post-Newtonian/Minkowskian theory in harmonic gauge
- separate gravitational from material degrees of freedom by imposing Lorenz gauge (or other wave gauge) on the total perturbation:

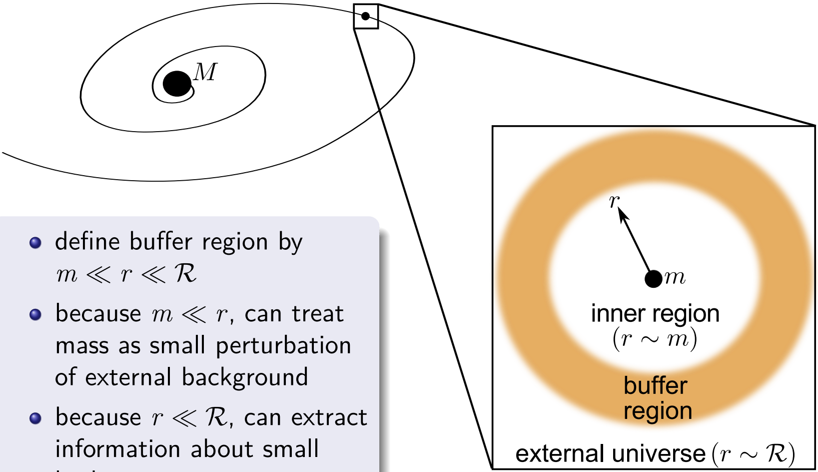
$$\nabla_\mu \bar{h}^{\mu\nu} = 0$$

- $\delta G_{\mu\nu}$ becomes a wave operator and EFE outside body becomes weakly nonlinear wave equation:

$$\Box \bar{h}_{\mu\nu} + 2R_\mu{}^\rho{}_\nu{}^\sigma \bar{h}_{\rho\sigma} = 2\delta^2 G_{\mu\nu}[h] + \ldots$$

- split into wave equation for each subsequent $h^{(n)}_{\mu\nu}[\gamma]$ and exactly solve with arbitrary γ
- gauge condition constrains γ (and other matter degrees of freedom)

Matched asymptotic expansions

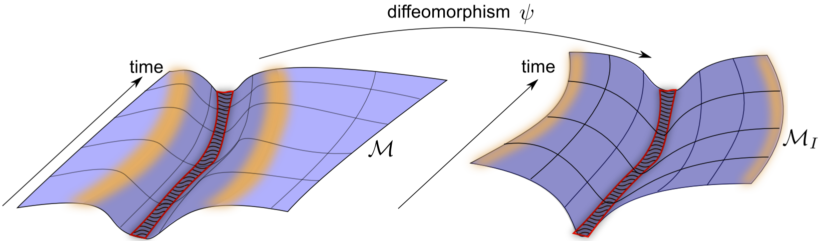


- define buffer region by $m \ll r \ll \mathcal{R}$
- because $m \ll r$, can treat mass as small perturbation of external background
- because $r \ll \mathcal{R}$, can extract information about small body

Matched asymptotic expansions: the *inner expansion*

Zoom in on body

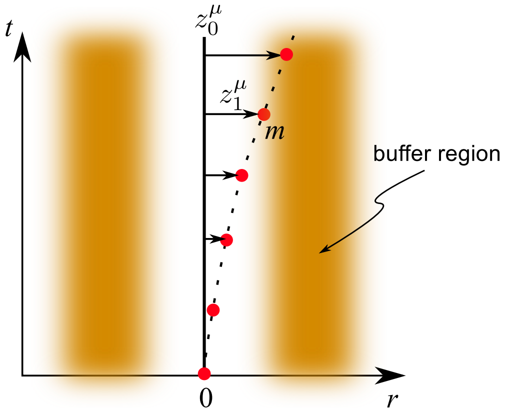
- use scaled coords $\tilde{r} \sim r/\epsilon$ to keep size of body fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed body defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $r \gg m$
  $\Rightarrow$ can define multipole moments without integrals over body



Position at first order: Gralla-Wald definition

Reminder: mass dipole moment

corresponds to displacement of center of mass from origin of coordinates



- work in coordinates centered on z_0^μ
- calculate mass dipole M^μ of inner background $g_{I\mu\nu}$
- first-order correction due to self-force:

$$mz_1^\mu \equiv M^\mu$$

Position at first order: self-consistent definition

Mass dipole about z^μ

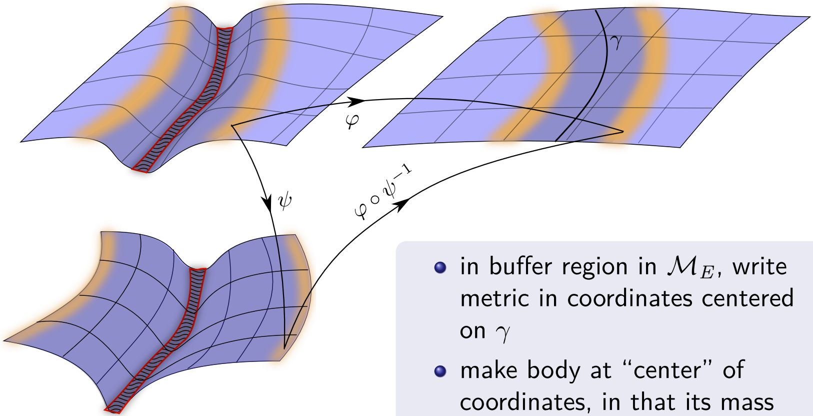
We want to find worldline z^μ for which $M^\mu = 0$

- work in coordinates centered on unspecified z^μ
- calculate mass dipole M^μ of inner background $g_{I\mu\nu}$
- first-order acceleration of z^μ: whatever ensures $M^\mu \equiv 0$

Position in self-consistent expansion (continued)

Enforce a relationship between the expansions

...to define a worldline for all time, even for black hole



- in buffer region in $\mathcal{M}_E$, write metric in coordinates centered on γ
- make body at "center" of coordinates, in that its mass dipole vanishes in $\mathcal{M}_I$

Outline

Solving the EFE in buffer region

Expansion for small r

- in coordinates centered on γ, allow all negative powers of r in $h^{(n)}_{\mu\nu}$
- but inner expansion must not have negative powers of ϵ
  $\Rightarrow$ most singular power of r in $\epsilon^n h^{(n)}_{\mu\nu}$ is $\frac{\epsilon^n}{r^n} = \frac{\epsilon^n}{\epsilon^n \tilde{r}^n} = \frac{1}{\tilde{r}^n}$

Therefore

$$h^{(n)}_{\mu\nu} = \frac{1}{r^n} h^{(n,-n)}_{\mu\nu} + r^{-n+1} h^{(n,-n+1)}_{\mu\nu} + r^{-n+2} h^{(n,-n+2)}_{\mu\nu} + \ldots$$

Information from inner expansion

- $1/\tilde{r}^n$ terms arise from asymptotic expansion of zeroth-order background in inner expansion
  $\Rightarrow$ $h^{(n,-n)}_{\mu\nu}$ is determined by multipole moments of isolated body

Form of solution in buffer region

What appears in the solution?

- put expansion into nth-order wave equation, solve order by order in r
- expand each $h^{(n,p)}_{\mu\nu}$ in spherical harmonics
- given a worldline γ, the solution at all orders is fully characterized by
  1. body's multipole moments (and corrections thereto): $\sim \frac{Y^{\ell m}}{r^{\ell+1}}$
  2. smooth solutions to vacuum wave equation: $\sim r^{\ell}\, Y^{\ell m}$
- everything else made of (linear or nonlinear) combinations of the above

Self field and regular field

- multipole moments define $h^{\mathrm{S}(n)}_{\mu\nu}$; interpret as bound field of body
- smooth homogeneous solutions define $h^{\mathrm{R}(n)}_{\mu\nu}$; free radiation, determined by global boundary conditions

First and second order solutions

First order

- $h^{(1)}_{\mu\nu} = h^{S(1)}_{\mu\nu} + h^{R(1)}_{\mu\nu}$
- $h^{S(1)}_{\mu\nu} \sim 1/r + O(r^0)$ defined by mass monopole m
- $h^{R(1)}_{\mu\nu}$ is undetermined homogenous solution regular at $r = 0$
- evolution equations (from gauge condition): $\dot{m} = 0$ and $a^\mu_{(0)} = 0$ (assuming $a^\mu = a^\mu_{(0)} + \epsilon a^\mu_{(1)} + \ldots$)

Second order

- $h^{(2)}_{\mu\nu} = h^{S(2)}_{\mu\nu} + h^{R(2)}_{\mu\nu}$
- $h^{S(2)}_{\mu\nu} \sim 1/r^2 + O(1/r)$ defined by
  1. monopole correction δm
  2. mass dipole M^μ
  3. spin dipole S^μ
- evolution equations: $\dot{S}^\mu = 0$, $\delta\dot{m} = \ldots$, and $\dot{M}^\mu = \ldots$

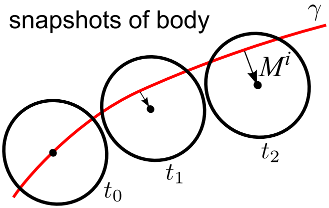
A master equation of motion

Evolution of mass dipole

$$\ddot{M}^\alpha - R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma M^\delta = -m a^\alpha_{(1)} + \tfrac{1}{2}R^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta} - \tfrac{1}{2}m\left(g^{\alpha\delta}+u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma$$

Includes

- geodesic deviation
- first-order term in acceleration of γ
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between a^α and M^α is valid for *any* γ



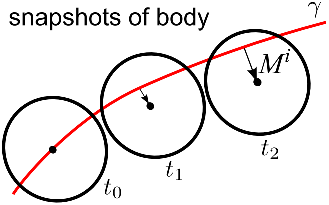
A master equation of motion

Evolution of mass dipole

$$\ddot{M}^\alpha - R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma M^\delta = -m a^\alpha_{(1)} + \tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} - \tfrac{1}{2} m \left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right) u^\beta u^\gamma$$

Includes

- geodesic deviation
- first-order term in acceleration of γ
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between a^α and M^α is valid for *any* γ



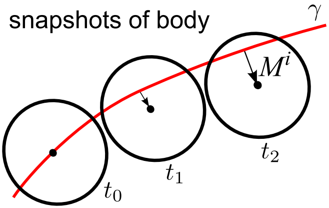
A master equation of motion

Evolution of mass dipole

$$\ddot{M}^\alpha - R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma M^\delta = -\boxed{m a^\alpha_{(1)}} + \tfrac{1}{2}R^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta} - \tfrac{1}{2}m\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma$$

Includes

- geodesic deviation
- first-order term in acceleration of γ
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between a^α and M^α is valid for *any* γ



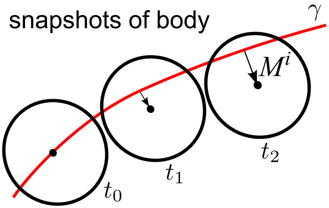
A master equation of motion

Evolution of mass dipole

$$\ddot{M}^\alpha - R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma M^\delta = -m a^\alpha_{(1)} + \boxed{\tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta}} - \tfrac{1}{2} m \left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right) u^\beta u^\gamma$$

Includes

- geodesic deviation
- first-order term in acceleration of γ
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between a^α and M^α is valid for *any* γ



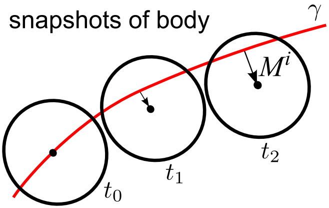
A master equation of motion

Evolution of mass dipole

$$\ddot{M}^\alpha - R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma M^\delta = -m a^\alpha_{(1)} + \tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} - \tfrac{1}{2} m \left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right) u^\beta u^\gamma$$

Includes

- geodesic deviation
- first-order term in acceleration of γ
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between a^α and M^α is valid for *any* γ



Equations of motion

Self-force in self-consistent expansion

- γ defined by $M^\alpha(t) \equiv 0$. Therefore

$$a^\alpha_{(1)} = -\tfrac{1}{2}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma + \tfrac{1}{2m}R^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta}$$

- through order ϵ, small body moves on a geodesic of $g_{\mu\nu} + h^R_{\mu\nu}$

Self-force in Gralla-Wald expansion

- γ is geodesic, so $a^\mu_{(n)} = 0$. Therefore

$$\frac{D^2 M^\alpha}{d\tau^2} = R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma M^\delta - \tfrac{1}{2}m\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta}\right)u^\beta u^\gamma + \tfrac{1}{2}R^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta}$$

Outline

Why second order?

Modeling EMRIs

- inspiral occurs very slowly, on radiation-reaction time $t_{rr} \sim 1/m$
- neglecting second-order self-force leads to error in acceleration $\delta a^\mu \sim m^2$
  $\Rightarrow$ error in position $\delta z^\mu \sim m^2 t^2$
  $\Rightarrow$ after radiation-reaction time $t_{rr} \sim 1/m$, error $\delta z^\mu \sim 1$

$\therefore$ accurately describing orbital evolution requires second-order force

Modeling IMRIs and similar-mass binaries

- second-order self-force should yield highly accurate model for IMRIs
- will fix terms quadratic in mass in post-Newtonian and Effective One Body theory

Position at second order: mass-centered gauges

Problem

- mass dipole moment defined for asymptotically flat spacetimes
- beyond zeroth order, inner expansion is not asymptotically flat

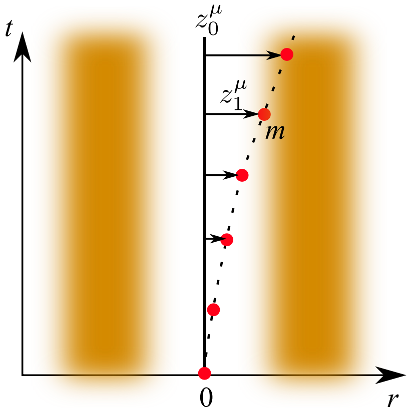
Solution

- find gauge in which field is manifestly mass-centered on z_0^μ (or z^μ)
- define position in other gauges by referring to transformation to that mass-centered gauge

Position at second order: Gralla's definition [2012]

Gauge in a Gralla-Wald-type expansion

On short timescales, position relative to z_0^μ is pure gauge



- start in gauge mass-centered on z_0^μ $\Rightarrow z_1^\mu = z_2^\mu = 0$
- under a small coordinate transformation, the position transforms just as coordinates do
- First order:

$$z_1^\mu = \xi_1^\mu|_{z_0}$$

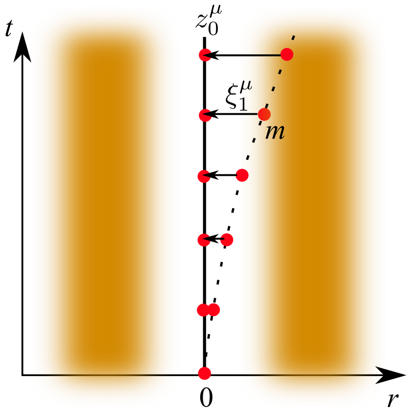
- Second order:

$$z_2^\mu = \xi_2^\mu|_{z_0} + \xi_1^\nu \partial_\nu \xi_1^\mu|_{z_0}$$

Position at second order: Gralla's definition [2012]

Gauge in a Gralla-Wald-type expansion

On short timescales, position relative to z_0^μ is pure gauge

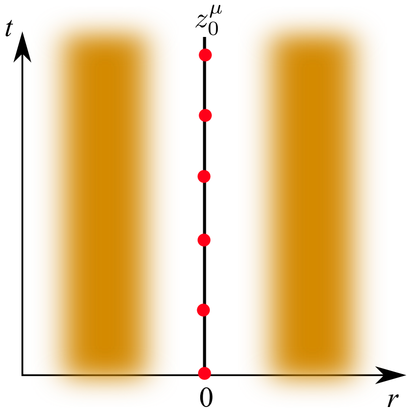


- start in gauge mass-centered on z_0^μ $\Rightarrow z_1^\mu = z_2^\mu = 0$
- under a small coordinate transformation, the position transforms just as coordinates do
- First order:
$$z_1^\mu = \xi_1^\mu|_{z_0}$$
- Second order:
$$z_2^\mu = \xi_2^\mu|_{z_0} + \xi_1^\nu \partial_\nu \xi_1^\mu|_{z_0}$$

Position at second order: Gralla's definition [2012]

Gauge in a Gralla-Wald-type expansion

On short timescales, position relative to z_0^μ is pure gauge

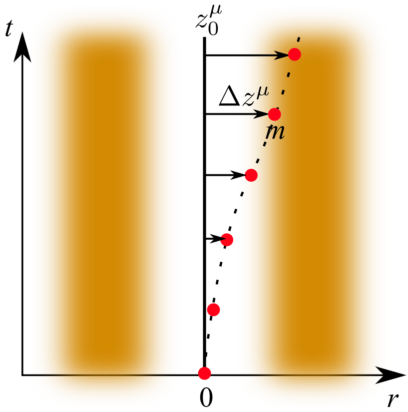


- start in gauge mass-centered on z_0^μ $\Rightarrow z_1^\mu = z_2^\mu = 0$
- under a small coordinate transformation, the position transforms just as coordinates do
- First order:
$$z_1^\mu = \xi_1^\mu|_{z_0}$$
- Second order:
$$z_2^\mu = \xi_2^\mu|_{z_0} + \xi_1^\nu \partial_\nu \xi_1^\mu|_{z_0}$$

Position at second order: Gralla's definition [2012]

Gauge in a Gralla-Wald-type expansion

On short timescales, position relative to z_0^μ is pure gauge

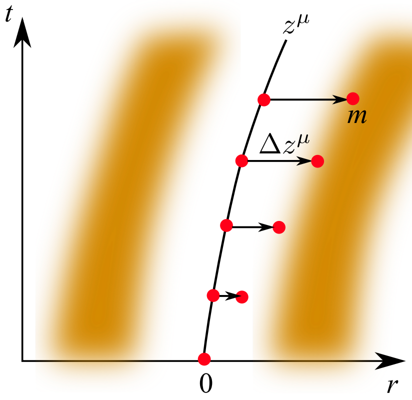


- start in gauge mass-centered on z_0^μ $\Rightarrow z_1^\mu = z_2^\mu = 0$
- under a small coordinate transformation, the position transforms just as coordinates do
- First order:
$$z_1^\mu = \xi_1^\mu|_{z_0}$$
- Second order:
$$z_2^\mu = \xi_2^\mu|_{z_0} + \xi_1^\nu \partial_\nu \xi_1^\mu|_{z_0}$$

Position at second order: self-consistent [Pound '12]

Gauge in a self-consistent expansion

Over a radiation-reaction time, position relative to z_0^μ is *not* pure gauge



- start in gauge mass-centered on z^μ
- demand that transformation to practical (e.g., Lorenz) gauge does not move z^μ
- i.e., insist
  $$\lim_{r\to 0}\int \xi^a_{(n)}\, d\Omega = 0$$
- ensures worldline in the two gauges is the same

Self-consistent equation of motion in Lorenz gauge

Neglecting object's spin and quadrupole moment,

$$\frac{D^2 z^\mu}{d\tau^2} = \frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\mathrm{R}\rho}_\nu\right)\left(h^{\mathrm{R}}_{\sigma\lambda;\rho} - 2h^{\mathrm{R}}_{\rho\sigma;\lambda}\right)u^\sigma u^\lambda + O(\epsilon^3)$$

- here $h^{\mathrm{R}}_{\mu\nu} = \epsilon h^{\mathrm{R}(1)}_{\mu\nu} + \epsilon^2 h^{\mathrm{R}(2)}_{\mu\nu}$

Generalized equivalence principle

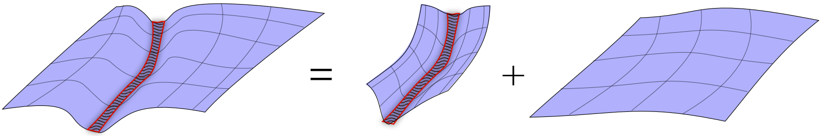
- z^μ satisfies geodesic equation in $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$
- recall: here $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$ is a "physical" field in the sense of satisfying vacuum EFE
- extends results of Detweiler-Whiting to second order

Outline

Effective interior metric

From self-field to singular field

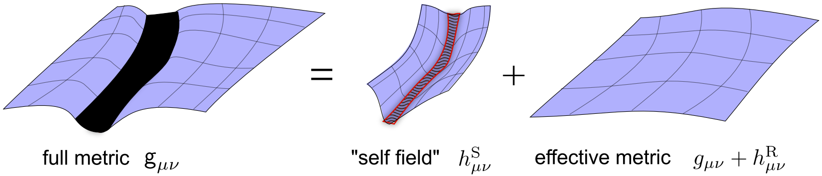
- $h^{\mathrm{S}}_{\mu\nu}$ and $h^{\mathrm{R}}_{\mu\nu}$ derived only in buffer region
- simply extend them to all $r > 0$ (and $r = 0$, for $h^{\mathrm{R}}_{\mu\nu}$)
- does not change field in buffer region or beyond



Effective interior metric

From self-field to singular field

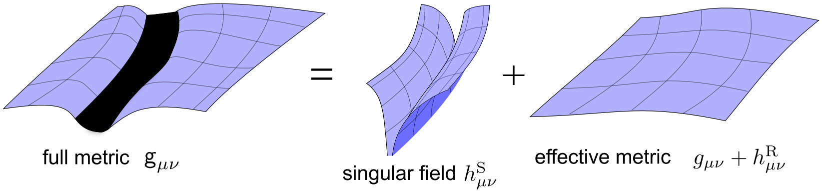
- $h^{\rm S}_{\mu\nu}$ and $h^{\rm R}_{\mu\nu}$ derived only in buffer region
- simply extend them to all $r > 0$ (and $r = 0$, for $h^{\rm R}_{\mu\nu}$)
- does not change field in buffer region or beyond



Effective interior metric

From self-field to singular field

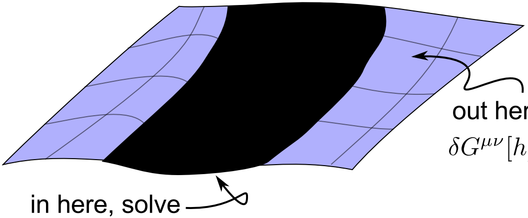
- $h^{\mathrm{S}}_{\mu\nu}$ and $h^{\mathrm{R}}_{\mu\nu}$ derived only in buffer region
- simply extend them to all $r>0$ (and $r=0$, for $h^{\mathrm{R}}_{\mu\nu}$)
- does not change field in buffer region or beyond



Obtaining global solution

Puncture/effective source scheme

- define $h^{\mathcal{P}}_{\mu\nu}$ as small-r expansion of $h^{\mathrm{S}}_{\mu\nu}$ truncated at finite order in r
- define $h^{\mathcal{R}}_{\mu\nu} = h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu} \simeq h^{\mathrm{R}}_{\mu\nu}$



out here, solve

$$\delta G^{\mu\nu}[h^{(2)}_{\rho\sigma}] = -\delta^2 G^{\mu\nu}[h^{(1)}_{\rho\sigma}]$$

in here, solve

$$\delta G^{\mu\nu}[h^{\mathcal{R}(2)}_{\rho\sigma}] = -\delta^2 G^{\mu\nu}[h^{(1)}_{\rho\sigma}] - \delta G^{\mu\nu}[h^{\mathcal{P}(2)}_{\rho\sigma}]$$

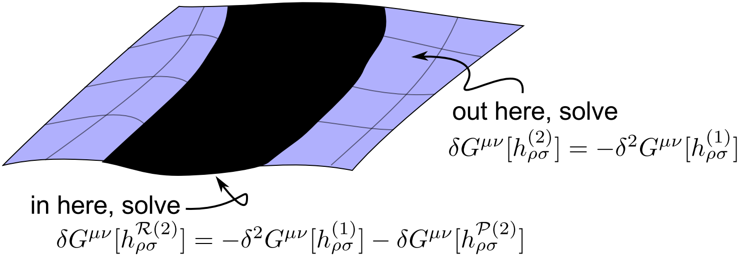
The point...

- to calculate effective metric "inside" body and full metric everywhere else, all you need is $h^{\mathrm{S}}_{\mu\nu}$ found in buffer region

Obtaining global solution

Puncture/effective source scheme

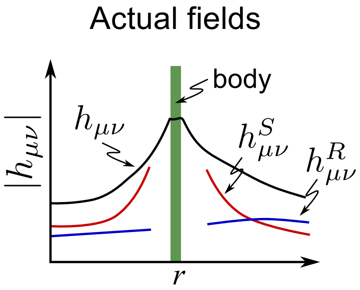
- define $h^{\mathcal{P}}_{\mu\nu}$ as small-r expansion of $h^{\rm S}_{\mu\nu}$ truncated at finite order in r
- define $h^{\mathcal{R}}_{\mu\nu} = h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu} \simeq h^{\rm R}_{\mu\nu}$



The point...

- to calculate effective metric "inside" body and full metric everywhere else, all you need is $h^{\rm S}_{\mu\nu}$ found in buffer region

More on puncturing



A note on singularities

- derivations of self-force from matched expansions yield an expression for the force in terms of a manifestly finite field outside the object
- we don't begin with an infinity and subtract an infinity
  —we write a known finite field as the limit of the difference between two known divergent fields

Effective stress-energy tensor

What does the field look like outside the object?

- recall multipole moment terms $\sim Y^{\ell m}/r^{\ell+1}$ in $h^{\mathrm{S}}_{\mu\nu}$
- using distribution theory, can show these terms (*not* entire field) are *effectively* sourced by δ functions and derivatives of them on γ
  $\Rightarrow$ in buffer region and further away, body looks like a skeleton of multipole moments on γ

Point particle picture recovered

- at first order, *entire* field identical to one that is sourced by point-particle stress-energy $T^{\mu\nu}_{(1)}[\gamma] = \int_\gamma m u^\mu u^\nu \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$
- all the early point-particle results hold true

Effective stress-energy tensor

What does the field look like outside the object?

- recall multipole moment terms $\sim Y^{\ell m}/r^{\ell+1}$ in $h^{\rm S}_{\mu\nu}$
- using distribution theory, can show these terms (*not* entire field) are *effectively* sourced by δ functions and derivatives of them on γ
  $\Rightarrow$ in buffer region and further away, body looks like a skeleton of multipole moments on γ

Point particle picture recovered

- at first order, *entire* field identical to one that is sourced by point-particle stress-energy $T^{\mu\nu}_{(1)}[\gamma] = \int_\gamma m u^\mu u^\nu \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$
- **all the early point-particle results hold true**

Conclusion

Determining the motion of a small body

- matched asymptotic expansions and self-consistent expansion used to overcome problems of multiple scales
- a self-gravitating compact object moves as a test body in an effective geometry $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$
- self-field $h^{\mathrm{S}}_{\mu\nu}$ calculated in buffer region outside body suffices to determine both $h^{\mathrm{R}}_{\mu\nu}$ and $h_{\mu\nu}$

Current status

- for spherical, nonspinning body, analytical portion of problem now solved at second order [Pound 2012, Gralla 2012]
- wealth of numerical results at first order [Barack, Detweiler, etc.]
- numerical implementation of second-order scheme is underway
- for more general body, we will require some model for evolution of body's multipole moments

Singular field (spin terms only)

$$
\begin{aligned}
\bar{h}^{ta}_{(2)} = &-\frac{2\epsilon^{aij}S_j n_i}{r^2} + \frac{3\epsilon^{ajd}S_d a^i \hat{n}_{ij}}{r} + \left(\tfrac{1}{3}S^c\mathcal{E}_b{}^d\epsilon^a{}_{cd} - \tfrac{49}{15}S^c\mathcal{E}^{ad}\epsilon_{bcd}\right)n^b \\
&+ 3a^a a^b S^d\epsilon_{bcd}n^c + \tfrac{1}{3}S^b\mathcal{E}^{cd}\epsilon_{bc}{}^i\hat{n}^a{}_{di} - S^b\mathcal{E}^{cd}\epsilon^a{}_b{}^i\hat{n}_{cdi} + \tfrac{15}{4}a^b a^c S^d\epsilon^a{}_d{}^i\hat{n}_{bci} \\
&+ r\Big(\tfrac{1}{4}S^b\epsilon^a{}_{bc}a_{d,tt}\hat{n}^{cd} + \tfrac{1}{3}S^b\epsilon^a{}_b{}^c a_{c,tt} - \tfrac{1}{18}S^b\dot{\mathcal{B}}^{cd}\hat{n}^a{}_{bcd} - \tfrac{11}{63}S^a\dot{\mathcal{B}}^{bc}\hat{n}_{bc} \\
&+ \tfrac{34}{63}S^b\dot{\mathcal{B}}_b{}^c\hat{n}^a{}_c + \tfrac{41}{63}S^b\dot{\mathcal{B}}^{ac}\hat{n}_{bc} + \tfrac{16}{15}S^b\dot{\mathcal{B}}^a{}_b - \tfrac{1}{2}a^b S^c\mathcal{E}^{di}\epsilon_{cd}{}^j\hat{n}^a{}_{bij} \\
&+ \tfrac{5}{2}a^b S^c\mathcal{E}^{di}\epsilon^a{}_c{}^j\hat{n}_{bdij} + \tfrac{3}{7}a^a S^b\mathcal{E}^{cd}\epsilon_{bdi}\hat{n}_c{}^i + \tfrac{2}{21}a^b S^c\mathcal{E}^{di}\epsilon_{bci}\hat{n}^a{}_d \\
&+ \tfrac{3}{7}a^b S^c\mathcal{E}_b{}^d\epsilon_{cdi}\hat{n}^{ai} + \tfrac{13}{42}a^b S^c\mathcal{E}^{di}\epsilon^a{}_{ci}\hat{n}_{bd} + \tfrac{69}{14}a^b S^c\mathcal{E}^{ad}\epsilon_{cdi}\hat{n}_b{}^i \\
&+ \tfrac{17}{21}a^b S^c\mathcal{E}^{di}\epsilon^a{}_{bc}\hat{n}_{di} + \tfrac{25}{84}a^b S^c\mathcal{E}_b{}^d\epsilon^a{}_{ci}\hat{n}_d{}^i - \tfrac{7}{3}a^b S^c\mathcal{E}^{ad}\epsilon_{bci}\hat{n}_d{}^i \\
&+ \tfrac{2}{15}a^b S^c\mathcal{E}_b{}^d\epsilon^a{}_{cd} + \tfrac{191}{45}a^b S^c\mathcal{E}^{ad}\epsilon_{bcd} + \tfrac{1}{6}S^b\epsilon_{bc}{}^j\mathcal{E}^{cdi}\hat{n}^a{}_{dij} \\
&+ \tfrac{1}{6}S^b\epsilon^a{}_{bi}\mathcal{E}_{cd}{}^i\hat{n}^{cd} + \tfrac{61}{42}S^b\mathcal{E}^a{}_c{}^i\epsilon_{bdi}\hat{n}^{cd} - \tfrac{35}{8}a^b a^c a^d S^i\epsilon^a{}_i{}^j\hat{n}_{bcdj} \\
&+ \tfrac{15}{2}a^a a^b a^c S^d\epsilon_{cdi}\hat{n}_b{}^i - \tfrac{1}{4}S^b\epsilon^a{}_b{}^j\mathcal{E}^{cdi}\hat{n}_{cdij}\Big) + O(r^2),
\end{aligned}
$$

▸ Return