The light-cone averaging and the luminosity-redshift relation

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M. Gasperini, G.M., F. Nugier, G. Veneziano, JCAP 07 (2011) 008;
I. Ben-Dayan, M. Gasperini, G.M., F. Nugier, G. Veneziano,
e-Print: arXiv:1202.1247 [astro-ph.CO].

Outline

- Light-cone averaging: formalism and motivation
- Geodesic light-cone coordinates
- Luminosity distance in a perturbed FLRW geometry
- Backreaction on the luminosity-redshift relation via light-cone averaging
- Conclusions

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The Problem

The large scale properties of our Universe are usually described in the context of a homogeneous and isotropic FLRW space-time.

However:

The real Universe is not exactly homogeneous and isotropic

- neither in its present state (classic inhomogeneities)
- nor in its primordial state (quantum fluctuations)

Inhomogeneities could affect in a non-trivial way the cosmological evolution.

One needs a well defined averaging procedure for smoothing-out the perturbed (non-homogeneous) geometric parameters.
How to determine the averaging procedure and the true dynamical evolution of the averaged cosmological geometry?

Not obvious!

Light-cone averaging: motivation

A phenomenological reconstruction of the spacetime metric and of its dynamic evolution on a cosmological scale is necessarily based on past light-cone observations, since most of the relevant signals travel with the speed of light.

The averaging procedure should be so referred to a null hypersurface coinciding with the past light-cone of our observer.

Let us start with a four-dimensional integral on a region bounded by two hypersurfaces, one spacelike and the other one null

$$I(S; -; A_0, V_0) = \int d^4x \sqrt{-g}\, \Theta(V_0 - V)\Theta(A - A_0)S(x),$$

where $V(x)$ is a scalar satisfying $\partial_\mu V \partial^\mu V = 0$ (with $V(x) = V_0$ the past light-cone of the observer) and $A(x)$ a timelike scalar.

Starting with this hypervolume integral we can construct covariant and gauge invariant hypersurface and surface integrals considering the variation of the volume average along the flow lines n_μ normal to $\Sigma(A)$.

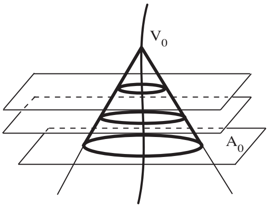
Light-cone averaging: prescription

(a) Considering the variation of the hypervolume integral by shifting the light-cone $V = V_0$ along the flow lines defined by n_μ, we obtain

$$I(S; V_0; A_0) = \int d^4x \sqrt{-g}\, \delta(V_0 - V)\Theta(A - A_0) \frac{|\partial_\mu V \partial^\mu A|}{\sqrt{-\partial_\nu A \partial^\nu A}} S(x)$$

which gives the integral on the past light-cone itself starting from a given hypersurface in the past.

truncated light cone



(a) $I(1; V_0; A_0)$

The averages of a scalar S is then defined by:
$$\langle S \rangle_{V_0}^{A_0} = \frac{I(S; V_0; A_0)}{I(1; V_0; A_0)}$$

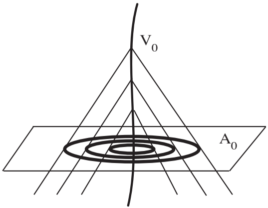
Light-cone averaging: prescription

(b) Considering the variation of the hypervolume integral by shifting the hypersurface $A = A_0$ along n_μ we obtain

$$I(S; A_0; V_0) = \int d^4x \sqrt{-g}\, \Theta(V_0 - V)\delta(A - A_0)\sqrt{-\partial_\mu A \partial^\mu A}\, S(x)$$

which gives the integral on the section of the hypersurface $A(x) = A_0$ which is causally connected with us.

causally connected sphere



(b) $I(1; A_0; V_0)$

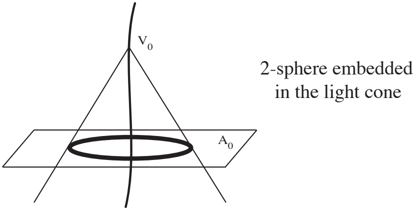
The averages of a scalar S is then defined by: $\langle S \rangle_{A_0}^{V_0} = \frac{I(S; A_0; V_0)}{I(1; A_0; V_0)}$

Light-cone averaging: prescription

(c) Considering the variation of the hypervolume integral both by shifting the light-cone $V = V_0$ and the hypersurface $A = A_0$ along the flow lines defined by n_μ we obtain

$$I(S; V_0, A_0; -) = \int d^4x \sqrt{-g}\, \delta(V_0 - V)\delta(A - A_0)|\partial_\mu V \partial^\mu A| S(x)$$

which gives the integral on the 2-sphere embedded in the past light-cone.



(c) $I(1; V_0, A_0; -)$

The averages of a scalar S is then defined by: $\langle S \rangle_{V_0, A_0} = \dfrac{I(S; V_0, A_0; -)}{I(1; V_0, A_0; -)}$

Buchert-Ehlers commutation rules on the light-cone

Starting from the definitions given we can obtain the corresponding gauge invariant generalizations of the Buchert-Ehlers commutation rule (Buchert, Ehlers (1997)), for example for the average $\langle S\rangle_{V_0,A_0}$ one obtains

$$\frac{\partial}{\partial A_0}\langle S\rangle_{V_0,A_0} = \left\langle \frac{k\cdot\partial S}{k\cdot\partial A}\right\rangle_{V_0,A_0} + \left\langle \frac{\nabla\cdot k}{k\cdot\partial A} S\right\rangle_{V_0,A_0} - \left\langle \frac{\nabla\cdot k}{k\cdot\partial A}\right\rangle_{V_0,A_0} \langle S\rangle_{V_0,A_0},$$

$$\begin{aligned}\frac{\partial}{\partial V_0}\langle S\rangle_{V_0,A_0} \quad = \quad & \left\langle \frac{\partial A\cdot\partial S}{k\cdot\partial A}\right\rangle_{V_0,A_0} - \left\langle k\cdot\partial S\frac{(\partial A)^2}{(k\cdot\partial A)^2}\right\rangle_{V_0,A_0} \\ & + \left\langle \left[\Box A - \nabla_\mu\left(k^\mu\frac{(\partial A)^2}{k\cdot\partial A}\right)\right]\frac{S}{k\cdot\partial A}\right\rangle_{V_0,A_0} \\ & - \left\langle \left[\Box A - \nabla_\mu\left(k^\mu\frac{(\partial A)^2}{k\cdot\partial A}\right)\right]\frac{1}{k\cdot\partial A}\right\rangle_{V_0,A_0} \langle S\rangle_{V_0,A_0},\end{aligned}$$

with $k^\mu \equiv \partial^\mu V$, $k^\mu\partial_\mu S = k\cdot\partial S$, $k^\mu\partial_\mu A = k\cdot\partial A$, $\partial_\mu A\partial^\mu S = \partial A\cdot\partial S$, $\partial_\mu A\partial^\mu A = (\partial A)^2$, $\nabla_\mu k^\mu = \nabla\cdot k$ and $\Box = \nabla^\mu\nabla_\mu$.

Physical Applications

Information about the large scale structure of our Universe reaches us travelling along the null geodesics of a possibly inhomogeneous spacetime.

Possible applications of the previous formalism are the averaging of the following quantities

- The redshift $1 + z = \frac{(k_\mu n^\mu)_s}{(k_\nu n^\nu)_0}$.
- The luminosity distance d_L.
- The so-called redshift drift.

which take a simpler form in a special ”adapted” coordinate system that we call ”geodesic light-cone coordinates (GLC)”.

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Geodesic light-cone coordinates

Such adapted coordinate system $x^\mu = (w, \tau, \tilde{\theta}^a)$, $a = 1, 2$ is defined by the metric:

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)\;; \qquad a, b = 1, 2\;.$$

This metric depends on six arbitrary functions (Υ, the two-dimensional vector U^a and the symmetric tensor γ_{ab}) and corresponds to a complete gauge fixing.
As it is easy to check w is a null coordinate while $\partial_\mu \tau$ defines a geodesic flow.

Let us underline that such coordinates can be seen as a particular specification of the "observational coordinates" (Maartens (1980) and Ellis, Nel, Maartens, Stoeger, Whitman (1985)).

To understand the geometric meaning of GLC coordinates let us consider the limiting case of a spatially flat FLRW Universe

$$w = r + \eta, \qquad \tau = t, \qquad \Upsilon = a(t), \qquad U^a = 0,$$
$$\gamma_{ab} d\theta^a d\theta^b = a^2(t) r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where η is the conformal time of the homogeneous metric: $d\eta = dt/a$.

Redshift to luminosity-distance relation

In the GLC gauge and in the case where the reference hypersurface $\Sigma(A)$ defines a geodesic observer ($V = w$ and $A = \tau$) the averaging on the 2-sphere embedded in the past light-cone takes the following semplified form

$$\langle S \rangle_{w_0, \tau_s} = \frac{\int_\Sigma d^4x \sqrt{-g}\, \delta(w - w_0) \delta(\tau - \tau_s) S(\tau, w, \tilde{\theta}^a)\, |\partial_\mu \tau \partial^\mu w|}{\int_\Sigma d^4x \sqrt{-g}\, \delta(w - w_0) \delta(\tau - \tau_s)\, |\partial_\mu \tau \partial^\mu w|}$$

$$= \frac{\int d^2\tilde{\theta} \sqrt{\gamma(w_0, \tau_s, \tilde{\theta}^a)}\, S(w_0, \tau_s, \tilde{\theta}^a)}{\int d^2\tilde{\theta} \sqrt{\gamma(w_0, \tau_s, \tilde{\theta}^a)}}, \qquad (1)$$

The redshift becomes

$$1 + z = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{(\partial^\mu w \partial_\mu \tau)_s}{(\partial^\mu w \partial_\mu \tau)_o} = \frac{\Upsilon(w_0, \tau_0, \tilde{\theta}^a)}{\Upsilon(w_0, \tau_s, \tilde{\theta}^a)}$$

where the subscripts “o” and “s” denote, respectively, a quantity evaluated at the observer and source space-time position.

Redshift to luminosity-distance relation

The luminosity-distance is given by

$$d_L = (1+z)^2 d_A$$

where d_A is the angular distance of the source as seen from the observer.

In a generic metric d_A can be defined by the following equation (Ellis, Nel, Maartens, Stoger, Whitman (1985), Vanderveld, Flanagan, Wasserman (2007))

$$\frac{d}{d\lambda}(\ln d_A) = \frac{\Theta}{2} = \frac{1}{2}\nabla_\mu k^\mu$$

with λ affine parameter along the ray trajectory.

In GLC gauge we have $k^\mu = (0, -1/\Upsilon, 0, 0)$, and the 2-sphere embedded in the light-cone is orthogonal to the photon momentum. The angular distance can therefore be determinate to be

$$d_A(\lambda) = \gamma^{1/4}(\lambda)\left(\sin\tilde{\theta}^1\right)^{-1/2}$$

Redshift to luminosity-distance relation

The redshift to luminosity-distance relation is then obtained averaging d_L on the two-sphere of constant redshift z_s embedded in the light-cone

$$\langle d_L \rangle_{w_0, z_s} = (1+z_s)^2 \frac{\int d^2\tilde{\theta}\, \gamma^{1/2}(w_0, \tau(z_s, w_0, \tilde{\theta}^a), \tilde{\theta}^b) d_A(w_0, \tau(z_s, w_0, \tilde{\theta}^a), \tilde{\theta}^b)}{\int d^2\tilde{\theta}\, \gamma^{1/2}(w_0, \tau(z_s, w_0, \tilde{\theta}^a), \tilde{\theta}^b)} .$$

where $\tau(z_s, w_0, \tilde{\theta}^a)$ is the solution of:

$$\frac{\Upsilon(w_0, \tau, \tilde{\theta}^a)}{\Upsilon_o} = \frac{1}{1+z} ,$$

Therefore the physically meaningful (covariant and gauge invariant) average reduces to averaging over an appropriate two-dimensional surface a scalar object which is non local, as the integrand itself will contain integrals along lightlike geodesic curves lying on the given null hypersurface.

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Luminosity distance in a perturbed FLRW geometry

Let us consider a perturbed conformally flat FLRW background to describe the inhomogeneities of our Universe at large scale.

In the well-known Newtonian gauge (NG) we have

$$g_{NG}^{\mu\nu} = a^{-2}(\eta)\,\mathrm{diag}\left(-1 + 2\Psi, 1 + 2\Psi, (1 + 2\Psi)\gamma_0^{ab}\right)$$

where $\gamma_0^{ab} = \mathrm{diag}\left(r^{-2}, r^{-2}\sin^{-2}\theta\right)$.

To use the previous results we have to re-express this metric in GLC form. We use

$$g_{GLC}^{\rho\sigma}(x) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu} g_{NG}^{\mu\nu}(y)$$

and impose the following boundary conditions

- Non-singular trasformation around the observer position at $r = 0$.
- The two-dimensional spatial sections $r =$ const are locally parametrized at the observer positions by standard spherical coordinates, i.e. $\tilde{\theta}^a(0) = \theta^a = (\theta, \phi)$.

Luminosity distance in a perturbed FLRW geometry

Using then the useful (zeroth-order) light-cone variables $\eta_\pm = \eta \pm r$, we obtain

$$
\begin{aligned}
\tau &= \int_{\eta_{in}}^{\eta} d\eta' a(\eta') \left[1 + \Psi(\eta', r, \theta^a)\right] \ , \\
w &= \eta_+ + \int_{\eta_+}^{\eta_-} dx \, \hat{\Psi}(\eta_+, x, \theta^a) \ , \\
\tilde{\theta}^a &= \theta^a + \frac{1}{2} \int_{\eta_+}^{\eta_-} dx \, \hat{\gamma}_0^{ab}(\eta_+, x, \theta^a) \int_{\eta_+}^{x} dy \, \partial_b \hat{\Psi}(\eta_+, y, \theta^a) \ ,
\end{aligned}
\tag{2}
$$

where $\hat{\Psi}$ and $\hat{\gamma}_0^{ab}$ are the Bardeen potential and the matrix γ_0^{ab} given as functions of η_+, η_-, and where η_{in} represents an early enough time when the perturbation (or better the integrand) was negligible.

Luminosity distance in a perturbed FLRW geometry

The non-trivial entries of the GLC metric are then given by

$$\Upsilon = a(\eta)\left[1 + \hat{\Psi}(\eta_+, \eta_+, \theta^a) - \int_{\eta_+}^{\eta_-} dx\, \partial_+ \hat{\Psi}(\eta_+, x, \theta^a)\right] + \int_{\eta_{in}}^{\eta} d\eta' a(\eta') \partial_r \Psi(\eta', r, \theta^a)$$

$$U^a = \frac{1}{2}\hat{\gamma}_0^{ab} \int_{\eta_+}^{\eta_-} dx\, \partial_b \hat{\Psi}(\eta_+, x, \theta^a) - \frac{1}{a(\eta)}\gamma_0^{ab} \int_{\eta_{in}}^{\eta} d\eta' a(\eta')\, \partial_b \Psi(\eta', r, \theta^a)$$
$$+\frac{1}{2}\int_{\eta_+}^{\eta_-} dx\ \partial_+ \left[\hat{\gamma}_0^{ab}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\ \partial_b \hat{\Psi}(\eta_+, y, \theta^a)\right]$$
$$-\frac{1}{2}\lim_{x\to\eta_+}\left[\hat{\gamma}_0^{ab}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\ \partial_b \hat{\Psi}(\eta_+, y, \theta^a)\right]$$

$$\gamma^{ab} = \frac{1}{a(\eta)^2}\left\{\left[1 + 2\Psi(\eta, r, \theta^a)\right]\gamma_0^{ab}\right.$$
$$\left. +\frac{1}{2}\left[\hat{\gamma}_0^{ac}\int_{\eta_+}^{\eta_-} dx\, \partial_c \left(\hat{\gamma}_0^{bd}(\eta_+, x, \theta^a)\int_{\eta_+}^{x} dy\, \partial_d \hat{\Psi}(\eta_+, y, \theta^a)\right) + a \leftrightarrow b\right]\right\}.$$

Luminosity distance in a perturbed FLRW geometry

The above transformations can be immediately applied to obtain an explicit expression for the redshift parameter z_s

$$1 + z_s = \frac{a(\eta_0)}{a(\eta_s)} \left[1 + J(z_s, \theta^a) \right],$$

where $J = I_+ - I_r$, and where:

$$I_+ = \int_{\eta_+^s}^{\eta_-^s} dx \, \partial_+ \hat{\Psi}(\eta_+^s, x, \theta^a) = \Psi_s - \Psi_o - 2 \int_{\eta_s}^{\eta_0} d\eta' \, \partial_r \Psi(\eta', \eta_0 - \eta', \theta^a),$$

$$I_r = \int_{\eta_{in}}^{\eta_s} d\eta' \frac{a(\eta')}{a(\eta_s)} \partial_r \Psi(\eta', \eta_0 - \eta_s, \theta^a) - \int_{\eta_{in}}^{\eta_0} d\eta' \frac{a(\eta')}{a(\eta_0)} \partial_r \Psi(\eta', 0, \theta^a) \, .$$

with $\eta_\pm^s = \eta_s \pm r_s$, $\Psi_s = \Psi(\eta_s, \eta_0 - \eta_s, \theta^a)$, $\Psi_o = \Psi(\eta_0, 0, \theta^a)$.

While the angular distance is given by

$$d_A(\lambda_s) = a_s r_s \left[1 - \Psi_s - J_2(z_s, \theta) \right] \; ,$$

where

$$J_2 = \frac{1}{\eta_0 - \eta_s} \int_{\eta_s}^{\eta_0} d\eta \, \frac{\eta - \eta_s}{\eta_0 - \eta} \left[\partial_\theta^2 + \cot\theta \, \partial_\theta + \sin^{-2}\theta \partial_\phi^2 \right] \Psi(\eta', \eta 0 - \eta', \theta^a)$$

Luminosity distance in a perturbed FLRW geometry

For the full explicit expression of the luminosity distance d_L at constant redshift what we need is the first-order expansion on the factor $a_s r_s \equiv a(\eta_s) r_s$ with respect to the constant parameter z_s which localizes the given light source on the past light-cone $w = w_0$ of our observer.

Starting from the zero-order solution $\eta_s^{(0)}$ of

$$\frac{a(\eta_s^{(0)})}{a_0} = \frac{1}{1+z_s}$$

we obtain

$$[a_s r_s](z_s, \theta^a) = a(\eta_s^{(0)}) \Delta\eta \left[1 + 2\Psi_{\rm av} + \left(1 - \frac{1}{\mathcal{H}_s \Delta\eta} \right) J(z_s, \theta^a) \right]$$

where $\Delta\eta = \eta_0 - \eta_s^{(0)}$ and

$$\Psi_{\rm av} = \frac{1}{\Delta\eta} \int_{\eta_s}^{\eta_0} d\eta' \, \Psi(\eta', \eta_0 - \eta', \theta^a)$$

Luminosity distance in a perturbed FLRW geometry

The (first-order, non-homogeneous, non-averaged) expression of d_L to first order in our perturbed background is so given by

$$\frac{d_L(z_s,\theta^a)}{(1+z_s)a_0\Delta\eta} \equiv \frac{d_L(z_s,\theta^a)}{d_L^{FLRW}(z_s)} = 1-\Psi(\eta_s,\eta_0-\eta_s,\theta^a)+2\Psi_{\rm av}+\left(1-\frac{1}{\mathcal{H}_s\Delta\eta}\right)J-J_2\,.$$

This is a general expression valid in any FLRW background.

Considering a CDM-dominated Universe we find full agreement with the result for the luminosity distance at constant redshift computed in Bonvin, Durrer, Gasparini (2006), modulo a term which can be written as $\vec{v}_0\cdot\hat{n}$.

Such a term gives a subleading contribution to the backreaction and can be neglected with no impact on our final results.

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Space-time and ensemble average

Considering a stochastic background of primordial perturbation and an esensemble average of the light-cone one, nontrivial effects can only be obtained from quadratic and higher-order perturbative corrections ($\overline{\Psi} = 0$, $\overline{\Psi^2} \neq 0$).

Let us take a typical average of a scalar S over the compact surface Σ (topologically equivalent to a two-sphere) embedded on the past light-cone $w = w_0$ at constant z_s

$$\langle S \rangle_\Sigma = \frac{\int_\Sigma d^2\mu\, S}{\int_\Sigma d^2\mu}$$

Expanding $d\mu$ and S in a perturbative series

$$d^2\mu = (d^2\mu)^{(0)}(1 + \mu^{(1)} + \mu^{(2)}), \qquad S = S^{(0)}(1 + \sigma^{(1)} + \sigma^{(2)})$$

we obtain that

$$\overline{\langle S/S^{(0)} \rangle} = 1 + \overline{\langle \sigma_2 \rangle} + IBR_2 + \dots$$

where

$$IBR_2 = \overline{\langle \mu_1 \sigma_1 \rangle} - \overline{\langle \mu_1 \rangle \langle \sigma_1 \rangle}$$

is called induced backreaction term and depends only on first order perturbation.

Cosmic variance

The variance descibes the distribution of the values of $S/S^{(0)}$ around its mean value $\overline{\langle S/S^{(0)}\rangle}$.

This dispersion is due to both the fluctuation on the averaging surface and to those due to *ensemble* fluctuations. Let us thus define:

$$\mathrm{Var}[S/S^{(0)}] \equiv \overline{\left\langle \left(S/S^{(0)} - \overline{\langle S/S^{(0)}\rangle} \right)^2 \right\rangle} = \overline{\langle (S/S^{(0)})^2 \rangle} - \left(\overline{\langle S/S^{(0)}\rangle} \right)^2$$

which gives

$$\mathrm{Var}[S/S^{(0)}] = \overline{\langle \sigma_1^2 \rangle}$$

On the other hand, considering the dispersion of the angular average $\langle S/S^{(0)}\rangle$ due to the stochastic fluctuations, one obtains

$$\mathrm{Var}'[S/S^{(0)}] \equiv \overline{\left(\langle S/S^{(0)}\rangle - \overline{\langle S/S^{(0)}\rangle} \right)^2} = \overline{\left(\langle S/S^{(0)}\rangle \right)^2} - \left(\overline{\langle S/S^{(0)}\rangle} \right)^2$$

which gives

$$\mathrm{Var}'[S/S^{(0)}] = \overline{\langle \sigma_1 \rangle^2}$$

As we will see, such a quantity is much smaller than the previous one. The main reason for the dispersion lies in the angular scatter of the data rather than in their stochastic distribution due to the *ensemble*.

Luminosity distance-redshift relation

Let us consider $S = d_L(z_s, \theta^a)$ and calculate the impact of the inhomogeneities on the luminosity distance-redshift relation.
If we define

$$A_1 = -\Psi_s \quad , \quad A_2 = 2\Psi_{\rm av} \quad , \quad A_3 = \left(1 - \frac{1}{\mathcal{H}_s \Delta\eta}\right) I_+$$

$$A_4 = -\left(1 - \frac{1}{\mathcal{H}_s \Delta\eta}\right) I_r \quad , \quad A_5 = -J_2$$

it is straightforward to see that

$$\sigma_1 = A_1 + A_2 + A_3 + A_4 + A_5$$

and

$$\mu_1 = 2(A_1 + A_2 + A_3 + A_4)$$

We can now obtain the second-order induced backreaction IBR_2, and the variance of $d_L/d_L^{\rm FLRW}$.

Induced backreaction and dispersion

To implement the *ensemble* average of our stochastic background of scalar perturbations we expand Ψ as

$$\Psi(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{i\vec{k}\cdot\vec{x}} \Psi_k(\eta) E(\vec{k}) \,,$$

where E is a unit random variable satisfying $E^*(\vec{k}) = E(-\vec{k})$ as well as the *ensemble*-average condition:

$$\overline{E(\vec{k}_1)E(\vec{k}_2)} = \delta(\vec{k}_1 + \vec{k}_2).$$

According to this we obtain

$$\overline{\langle \Psi_s \Psi_s \rangle} = \int_0^\infty \frac{dk}{k}\, P_\Psi(k, \eta_s),$$

$$\overline{\langle \Psi_s \rangle \langle \Psi_s \rangle} = \int_0^\infty \frac{dk}{k}\, P_\Psi(k, \eta_s) \left(\frac{\sin(k\Delta\eta)}{k\Delta\eta} \right)^2$$

where

$$P_\Psi(k, \eta) \equiv \frac{k^3}{2\pi^2} |\Psi_k(\eta)|^2$$

Induced backreaction and dispersion

We have many similar contributions appearing both in the induced backreaction and in the variance.
Such contributions are generated from the A_i terms and can be parameterized, considering a time-independent Ψ_k (see next slide!), as

$$\overline{\langle A_i A_j \rangle} = \int_0^\infty \frac{dk}{k} \, P_\Psi(k) \mathcal{C}_{ij}(k, \eta_0, \eta_s),$$

$$\overline{\langle A_i \rangle \langle A_j \rangle} = \int_0^\infty \frac{dk}{k} \, P_\Psi(k) \mathcal{C}_i(k, \eta_0, \eta_s) \, \mathcal{C}_j(k, \eta_0, \eta_s)$$

and we have that the induced backreaction is given by

$$\mathrm{IBR}_2 = \int_0^\infty \frac{dk}{k} \, P_\Psi(k) \sum_{i=1}^{4} \sum_{j=1}^{5} 2 \Big[\mathcal{C}_{ij}(k, \eta_0, \eta_s) - \mathcal{C}_i(k, \eta_0, \eta_s) \, \mathcal{C}_j(k, \eta_0, \eta_s) \Big]$$

while the dispersion by

$$\left(\mathrm{Var} \left[\frac{d_L}{d_L^{FLRW}} \right] \right)^{1/2} = \sqrt{\overline{\langle \sigma_1^2 \rangle}} = \left[\int_0^\infty \frac{dk}{k} \, P_\Psi(k) \sum_{i=1}^{5} \sum_{j=1}^{5} \mathcal{C}_{ij}(k, \eta_0, \eta_s) \right]^{1/2}$$

Power Spectrum $P_\Psi(k,\eta)$

Limiting ourselves to sub-horizon perturbations, and considering the standard CDM model, we can approximate Ψ_k with a time-independent value and simply obtain this by applying an appropriate, time-independent transfer function to the primordial (inflationary) spectral distribution

$$P_\Psi(k) = \left(\frac{3}{5}\right)^2 \Delta^2_{\mathcal{R}} T^2(k), \qquad \Delta^2_{\mathcal{R}} = A\left(\frac{k}{k_0}\right)^{n_s-1},$$

where $T(k)$ is a constant transfer function and $\Delta^2_{\mathcal{R}}$ is the primordial power spectrum of curvature perturbations.
From WMAP data we have the following approximate values

$$A = 2.45\times 10^{-9}, \qquad n_s = 0.96, \qquad k_0/a_0 = 0.002\,\mathrm{Mpc}^{-1}$$

while, from Eisenstein, Hu (1998), we have

$$T_0(q) = \frac{L_0}{L_0 + q^2 C_0(q)}, \quad L_0(q) = \ln(2e+1.8q), \quad C_0(q) = 14.2 + \frac{731}{1+62.5q}$$

where $q = \frac{k}{13.41 k_{\rm eq}}$ and $k_{\rm eq}$ is the scale corresponding to matter-radiation equality.
In the following we consider a CDM model with $a_0 = 1$, $\Omega_m = 1$, and we use $h \equiv H_0/(100\,\mathrm{km\,s^{-1}Mpc^{-1}}) = 0.7$. We then have $k_{\rm eq} \simeq 0.036\,\mathrm{Mpc}^{-1}$.

Induced backreaction and dispersion: leading terms

In a CDM model we can analytically compute the coefficients $\mathcal{C}_{ij}, \mathcal{C}_i$.

Let us then do some consideration

- The combination $\mathcal{C}_{ij} - \mathcal{C}_i\mathcal{C}_j$ will go at least as $\mathcal{O}(k^2\Delta\eta^2)$ in the IR limit, i.e. for $k\Delta\eta \ll 1$. We have subleading infrared contribution and we can safely fix our infrared cut-off to be $k = H_0$.
- The main contribution to IBR_2 and to the dispersion will come from the range $1/\Delta\eta \ll k \leq 2.5\,\mathrm{Mpc}^{-1}$. Where the transfer function is not yet decreasing as $\log k/k^2$.

In particular, we have only two leading contributions, given by the integrals which involve the coefficients $\mathcal{C}_{44}$ and $\mathcal{C}_{55}$. For such terms we have the following behaviour for $k\Delta\eta \gg 1$:

$$\begin{aligned}
\mathcal{C}_{44} &\simeq \left(1 - \frac{1}{\mathcal{H}_s\Delta\eta}\right)^2 (\eta_0^2 + \eta_s^2)\frac{k^2}{27} \\
\mathcal{C}_{55} &\simeq \frac{k^3\Delta\eta^3}{15}\mathrm{SinInt}(k\Delta\eta)
\end{aligned}$$

Induced backreaction and dispersion: leading terms

$\overline{\langle A_4 A_4 \rangle}$ $\Rightarrow$ gives a large contribution for $z_s \ll 1$.

$\overline{\langle A_5 A_5 \rangle}$ $\Rightarrow$ gives a large contribution for $z_s \gg 1$.

The contributions depend in principle on the UV cut-off k_{UV} eventually used to evaluate the integrals.

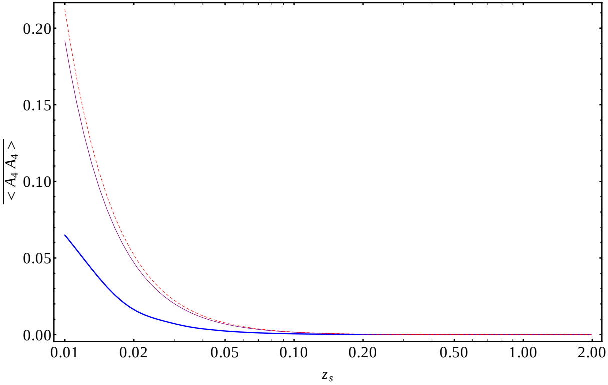
On the other hand, when k_{UV} is taken inside the regime where the spectrum goes like $(\log k)^2/k^4$, the dependence on the cut-off will be not too strong.

In particular, IBR_2 depends very weakly on the particular value of k_{UV}, while the dispersion has a somewhat stronger dependence on k_{UV}.

Both oh them are however finite when $k_{UV} \to +\infty$.

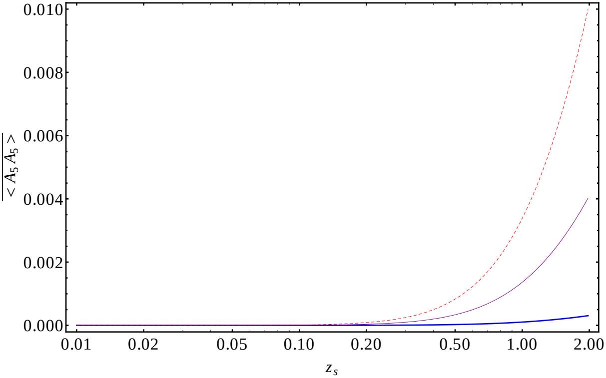
The numerical integrations of $\overline{\langle A_4 A_4 \rangle}$ and $\overline{\langle A_5 A_5 \rangle}$ are presented in the following two figures, where we illustrate the magnitude of the backreaction effect as a function of the redshift and of its (weak) dependence on the cut-off (ranging from $k_{UV} = 0.1\,\mathrm{Mpc}^{-1}$ to $k_{UV} = +\infty$).

$\overline{\langle A_4 A_4 \rangle}$ term



The result of the numerical integration for $\overline{\langle A_4 A_4 \rangle}$ is plotted as a function of z_s for three different values of the UV cut-off: $k = 0.1\,\mathrm{Mpc}^{-1}$ (thick blue line), $k = 1\,\mathrm{Mpc}^{-1}$ (thin purple line), $k = +\infty$ (dashed red line).

$\overline{\langle A_5 A_5 \rangle}$ term



The result of the numerical integration for $\overline{\langle A_5 A_5 \rangle}$ is plotted as a function of z_s for three different values of the UV cut-off: $k = 0.1\,\mathrm{Mpc}^{-1}$ (thick blue line), $k = 1\,\mathrm{Mpc}^{-1}$ (thin purple line), $k = +\infty$ (dashed red line).

Backreaction on the luminosity-redshift relation

Let us now sum up all contributions and compare the results of $\overline{\langle d_L \rangle} \pm d_L^{CDM} \sqrt{\overline{\langle \sigma_1^2 \rangle}}$ with the homogeneous luminosity-distance of a pure CDM model and of a successful ΛCDM model.

We will include into $\overline{\langle d_L \rangle}$ only the IBR_2 contribution, $\overline{\langle d_L \rangle} = d_L^{CDM}(1 + \mathrm{IBR}_2)$.

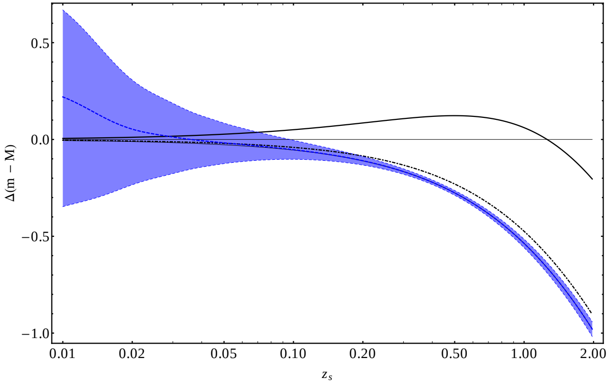
A full computation should include additional contributions arising from second-order perturbations of d_L. Nonetheless, our computation may estimate a reliable "lower limit" of the possible corrections to the luminosity-redshift relation in the context of our inhomogeneous geometry.

The comparison between the homogeneous and inhomogeneous (averaged) values of d_L can be conveniently illustrated by plotting the difference between the distance modulus of the considered model and that of a flat, linearly expanding Milne-type geometry

$$\Delta(m - M) = 5 \log_{10} \left[\overline{\langle d_L \rangle} \right] - 5 \log_{10} \left[\frac{(2 + z_s) z_s}{2 H_0} \right]$$

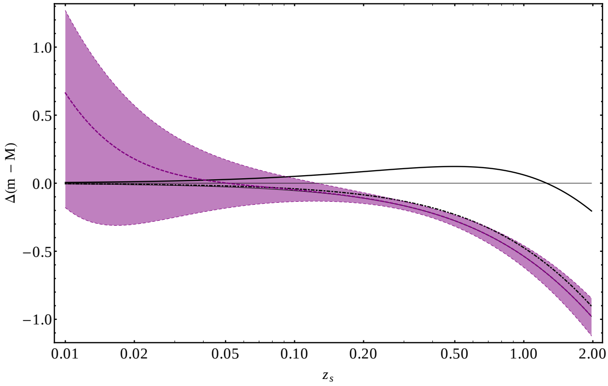
The results are illustrated in the following two figures for the case of a cut-off $k_{UV} = 0.1\,\mathrm{Mpc}^{-1}$ and $k_{UV} = 1\,\mathrm{Mpc}^{-1}$.

$\Delta(m - M)$ with cut-off $k_{UV} = 0.1\,\mathrm{Mpc}^{-1}$



The distance modulus is plotted for a pure CDM model (thin line), for a CDM model including the contribution of IBR_2 (dashed blue line) plus/minus the dispersion (coloured region), and for a ΛCDM model with $\Omega_\Lambda = 0.73$ (thick line) and $\Omega_\Lambda = 0.1$ (dashed-dot thick line).

$\Delta(m - M)$ with cut-off $k_{UV} = 1\,\mathrm{Mpc}^{-1}$



The distance modulus is plotted for a pure CDM model (thin line), for a CDM model including the contribution of IBR_2 (dashed blue line) plus/minus the dispersion (coloured region), and for a ΛCDM model with $\Omega_\Lambda = 0.73$ (thick line) and $\Omega_\Lambda = 0.1$ (dashed-dot thick line).

Backreaction on the luminosity-redshift relation

The choice of the cut-off may affect the final result when the values of k_{UV} are varying in the range $(0.1 - 1)\,\mathrm{Mpc}^{-1}$.

In any case the inhomogeneous model adopted is fully under control only in the linear perturbative regime. The spectrum cannot be extrapolated at scales higher than about $k \sim 1\,\mathrm{Mpc}^{-1}$ without taking into account the complicated effects of its non-linear dynamical evolution.

The corrections induced by IBR_2 on the luminosity distance of a homogeneous CDM model, even taking into account the expected dispersion of values around $\overline{\langle d_L \rangle}$, cannot be used to successfully simulate realistic dark-energy effects.

On the other hand, a consistent second-order computation of the backreaction should include the contribution of $\overline{\langle \sigma_2 \rangle}$.

Backreaction on the luminosity-redshift relation

It is then possible to show that $\overline{\langle \sigma_2 \rangle}$ contains contributions $\sim \overline{\langle A_5 A_5 \rangle}$.

The behaviour of this term, in the asymptotic regime $k\Delta\eta \gg 1$, is very different from the behaviour of terms like $\overline{\langle A_4 A_4 \rangle}$ which give the leading contribution to IBR_2: the contribution of $\overline{\langle A_5 A_5 \rangle}$, in particular, grows at large redshifts.

This suggests that a full computation of $\overline{\langle \sigma_2 \rangle}$ could strongly enhance the overall backreaction effects at large z_s, with respect to the effects due to IBR_2 discussed previously.

A full second-order computation, possibly joined to a reliable estimate of contributions from the non-linear regime, appears to be necessary before firm conclusions on the correct interpretation of the data can be drawn.

To conclude, the different behaviour of the different backreaction contributions, at small z_s and large z_s, represent an important signature to distinguish the effects due to averaged inhomogeneities from the more conventional dynamical effects of homogeneous dark energy sources.

Full second order by first order perturbation

The calculation of the full backreaction on the observed quantity requires a full second-order calculation of the flux associated to the supernovae luminosity ($\sim d_L^{-2}$).

On the other hand, suitable linear combinations of averages of different powers of d_L only depend of the first order quantity. As an example, one can show that the following equality holds at second order for any value of the real parameter α:

$$\overline{\langle (d_L/d_L^{FLRW})^{\alpha} \rangle} - \alpha \overline{\langle d_L/d_L^{FLRW} \rangle} = 1 - \alpha + \frac{\alpha(1-\alpha)}{2} \overline{\langle \sigma_1^2 \rangle}$$

This quantity can be plotted and compared with its (deterministic) value for ΛCDM, for various values of α.

The two models disagree for generic α, leading to the conclusion that realistic inhomogeneities added to CDM lead to a model that can be distinguished, in principle, from ΛCDM.

On th other hand, we only have a single quantity measured by the supernovae experiments (the flux), and one cannot exclude that the two models happen to give the same result for that particular observable.

Conclusions

- We have given a covariant and gauge invariant formalism to average on null hypersurfaces and to analyze the effects of inhomogeneities on astrophysical observables related to light-like (massless) signals.
- We have applied such formalism to evaluate the impact of the inhomogeneities on the luminosity-redshift relation.
- We obtain two leading backreaction contributions.
  $\overline{\langle A_4 A_4 \rangle}$, which is associated with "Doppler-type" contributions, and
  $\overline{\langle A_5 A_5 \rangle}$, which is associated with lensing contributions.
- Using a stochastic spectrum for the fluctuations we obtain a strong backreaction effect (estimated from $\overline{\langle A_5 A_5 \rangle}$) of the order of 10^{-3} for $z_s \sim 1$. Two order of magnitude more of what usually obtained in the literature.
- Depending on the full contributions of the second order and, in case, from the one of very small scales, the effects of inhomogeneities could be relevant for future precise determinations of the amount and equation of state of dark energy.

THANKS FOR THE ATTENTION!