Relativistic theory of tidal Love numbers and Tidal interaction of black holes

Taylor Binnington and Eric Poisson
Department of Physics
University of Guelph

Outline

I: Tidal Love numbers in general relativity

II: Tidal interaction of black holes versus Newtonian, viscous bodies

Epilogue: Breakdown of effacement principle at 4.5PN?

[Flanagan & Hinderer 2008; Damour & Nagar 2009; Damour & Lecian 2009]

I. Goals and motivation

Flanagan & Hinderer (2008):

Low-frequency (< 400 Hz) measurement of gravitational waves from neutron-star binary inspirals by enhanced LIGO will reveal (a weighted average of) the **tidal Love numbers** of both companions. This measurement can be used to constrain the neutron-star radius and the equation of state.

Hinderer (2008):

Love numbers computed in Regge-Wheeler gauge for polytropic equations of state.

We provide precise and gauge-invariant definitions of (electric-type and magnetic-type) Love numbers in general relativity; and we compute these for polytropes.
[Damour & Nagar]

I. Newtonian theory (1/2)

A spherical body of mass *M* and radius *R* is placed in a tidal field

$$U_{\rm tidal} = -\frac{1}{2}\mathcal{E}_{ab}x^a x^b; \qquad \mathcal{E}_{ab} = -\partial_{ab}U_{\rm ext}\Big|_{\rm body}$$

The body acquires a deformation; its own potential is

$$U_{\rm body} = \frac{M}{r} + \frac{3}{2}Q_{ab}\frac{x^a x^b}{r^5}$$

Dimensional analysis requires

$$Q_{ab} = -\frac{2}{3}k_2 R^5 \mathcal{E}_{ab}$$

Love number (k_2)

The total potential is

$$U = \frac{GM}{r} - \frac{1}{2}\left[1 + 2k_2(R/r)^5\right]\mathcal{E}_{ab}x^b x^b$$

I. Newtonian theory (2/2)

More generally, for a tidal field of multipole order *l*,

$$U = \frac{GM}{r} - \frac{1}{(\ell-1)\ell}\left[1 + 2k_\ell (R/r)^{2\ell+1}\right]\mathcal{E}_L x^L$$

$$\begin{aligned} \mathcal{E}_L x^L &= \mathcal{E}_{a_1 a_2 \cdots a_\ell} x^{a_1} x^{a_2} \cdots x^{a_\ell} \\ &= \sum_{m=-\ell}^{\ell} r^\ell \mathcal{E}_{\ell m} Y_{\ell m}(\theta, \phi) \end{aligned}$$

STF tensor

I. Relativistic theory (1/2)

We consider the vacuum exterior of a tidally-perturbed, spherical body.

The perturbed metric is calculated in **light-cone coordinates** that possess a clear geometrical meaning:

$$\begin{aligned} v: &\quad \text{constant on light cones converging toward } r=0 \\ r: &\quad \text{areal radius} \\ (\theta,\phi): &\quad \text{constant on generators} \end{aligned}$$

$$g_{vv} = -1 + \frac{2M}{r} - \frac{2}{(\ell-1)\ell}\Big[A(r) + 2k_{\rm el}(R/r)^{2\ell+1}B(r)\Big]\mathcal{E}_L x^L$$

$$\begin{aligned} A(r) &= (1-2M/r)^2 F(-\ell+2,-\ell;-2\ell;2M/r) \\ B(r) &= (1-2M/r)^2 F(\ell+1,\ell+3;2\ell+2;2M/r) \end{aligned}$$

I. Relativistic theory (2/2)

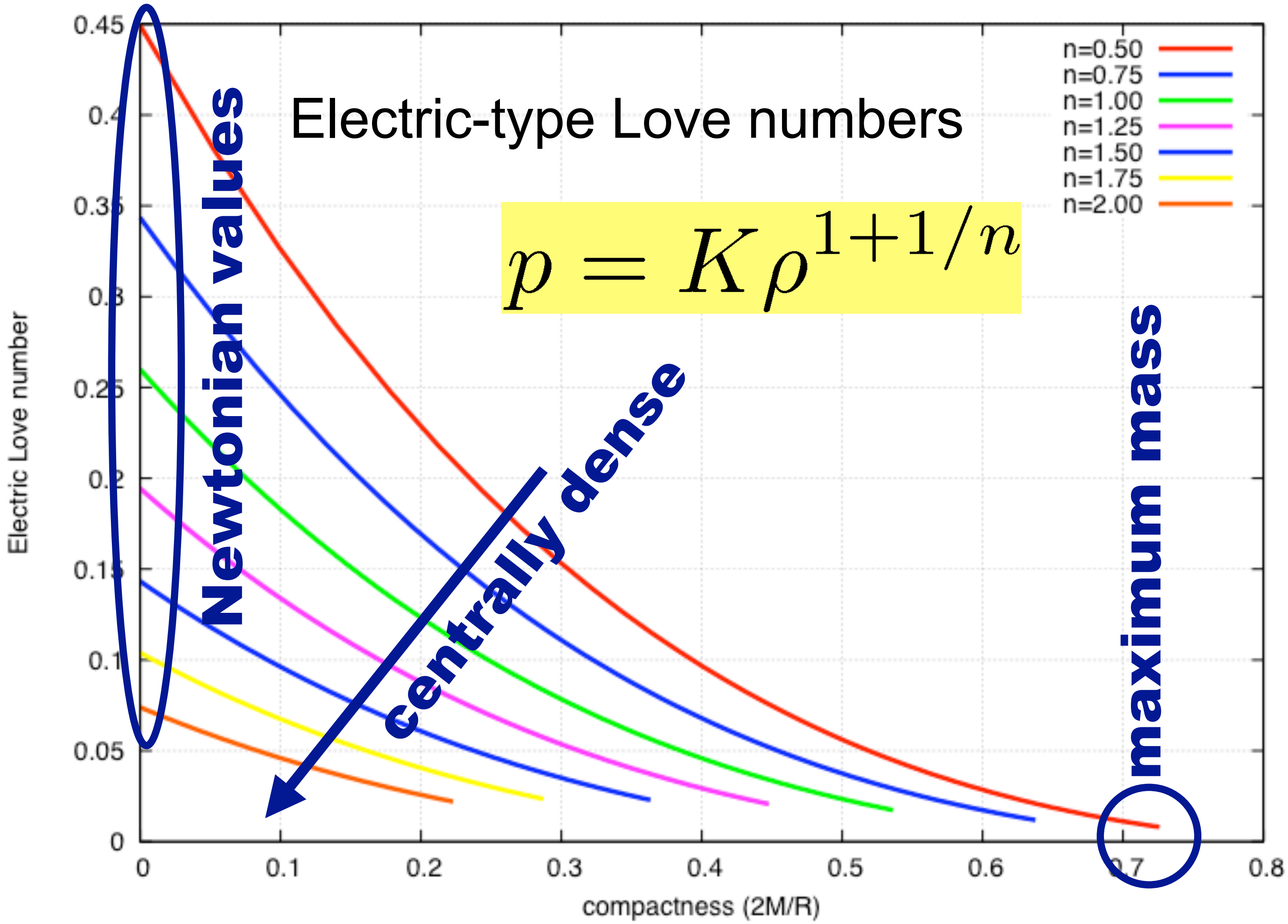
Here the tidal moments are defined in terms of the (electric-type) components of the **Weyl tensor** in the asymptotic region (*r* >> *R*).

The odd-parity sector of the perturbation involves other (the magnetic-type) components of the Weyl tensor, and a magnetic-type Love number.

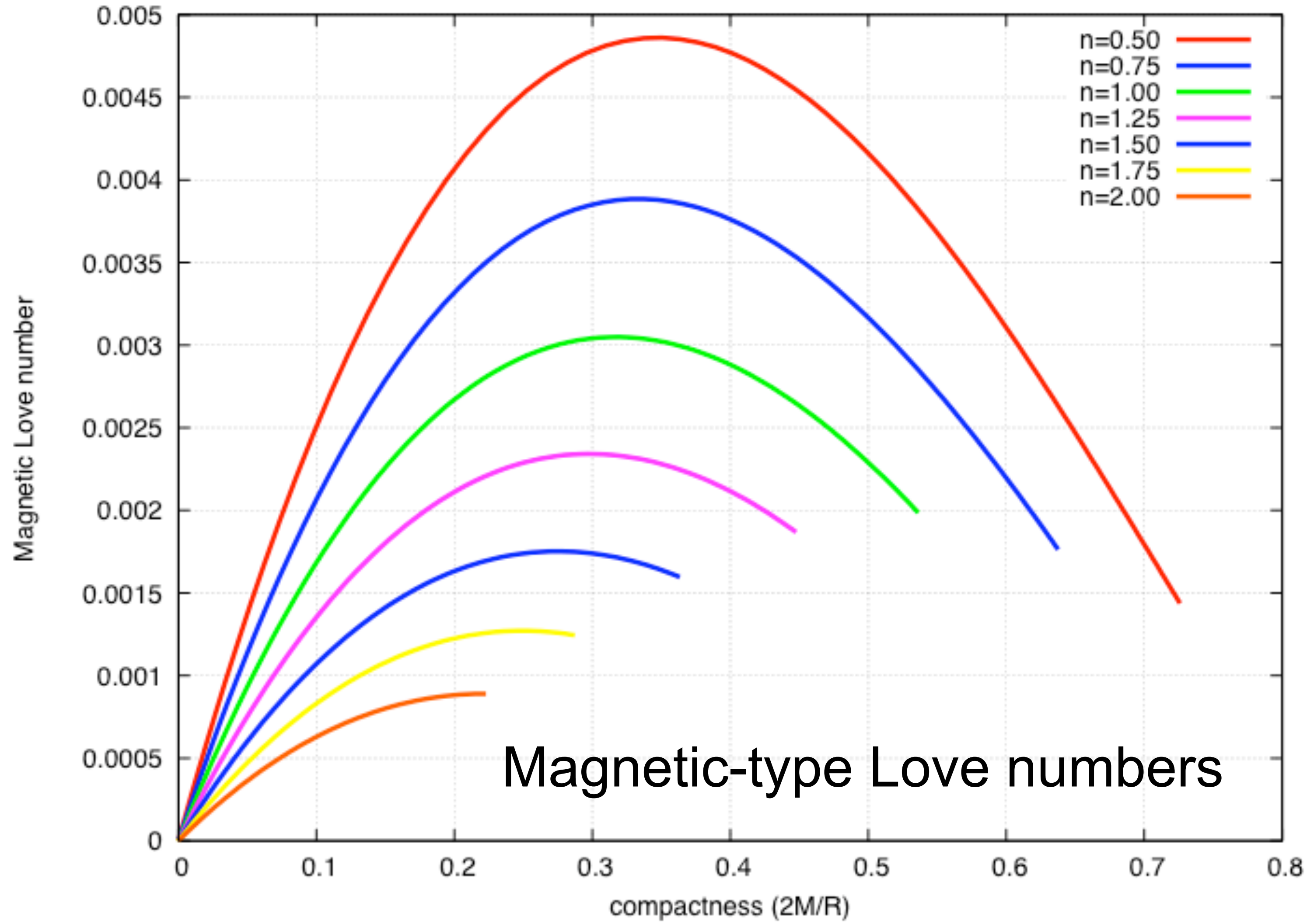
$$k_{\mathrm{el}} \qquad k_{\mathrm{mag}}$$

The metric perturbation can be presented in gauge-invariant variables: **the Love numbers are gauge-invariant.**

I. Computations (1/2)



I. Computations (2/2)



I. Black hole

The theory applies just as well to black holes (coordinates are well-behaved at the event horizon).

$$g_{vv} = -1 + \frac{2M}{r} - \frac{2}{(\ell-1)\ell}\Big[A(r) + 2k_{\rm el}(2M/r)^{2\ell+1}B(r)\Big]\mathcal{E}_L x^L$$

But *B* diverges logarithmically at the horizon; regularity requires that

The Love numbers of a black hole are all zero.

$$k_{\rm el} = 0 = k_{\rm mag}$$

This was observed previously [Poisson (2005); Fang & Lovelace (2005)], but never firmly articulated.

Outline

I: Tidal Love numbers in general relativity

II: Tidal interaction of black holes versus Newtonian, viscous bodies

II. Tides: Black hole

A **nonrotating** black hole in tidal interaction with nearby bodies undergoes a change of mass and angular momentum described by

$$\dot{M} = \frac{16}{45} M^6 \dot{\mathcal{E}}^{ab} \dot{\mathcal{E}}_{ab}, \qquad \dot{J} = -\frac{32}{45} M^6 \bigl(\epsilon^a{}_{cd} \mathcal{E}^{cb} s^d\bigr) \dot{\mathcal{E}}_{ab}$$

For a **rapidly-rotating** black hole we have $(\chi = J/M^2)$

$$\begin{aligned}\dot{J} \quad &= \quad -\frac{16}{45} M^6 \Omega_H \Bigl(1 + \sqrt{1-\chi^2}\Bigr) \Bigl[2(1+3\chi^2)\bigl(\mathcal{E}_{ab}\mathcal{E}^{ab}\bigr) \\ &\quad - 3\Bigl(1 + \frac{17}{4}\chi^2\Bigr)\bigl(\mathcal{E}_{ab} s^b \mathcal{E}^a{}_c s^c\bigr) + \frac{15}{4}\chi^2 \bigl(\mathcal{E}_{ab} s^a s^b\bigr)^2 \Bigr]\end{aligned}$$

II. Tides: Newtonian body (1/3)

These expressions admit a Newtonian interpretation.

A Newtonian, viscous body deformed by rotation and tidal forces has a quadrupole moment given by

$$Q_{jk} = \frac{2}{3} n_2 R^5 \Omega_H^2 C_{jk} - \frac{2}{3} k_2 R^5 \left(\mathcal{E}_{jk} - \tau \dot{\mathcal{E}}_{jk} \right)$$

rotation

tides

in the body's rotating frame.

$$\tau \propto \nu R/M$$
$$R = \text{body radius}$$

II. Tides: Newtonian body (1/3)

These expressions admit a Newtonian interpretation.

A Newtonian, viscous body deformed by rotation and tidal forces has a quadrupole moment given by

$$Q_{jk} = \frac{2}{3} n_2 R^5 \Omega_H^2 C_{jk} - \frac{2}{3} k_2 R^5 \left(\mathcal{E}_{jk} - \tau \dot{\mathcal{E}}_{jk} \right)$$

$$\mathcal{E}_{jk}(t - \tau)$$

viscous delay

$$\tau \propto \nu R / M$$

in the body’s rotating frame.

II. Tides: Newtonian body (1/3)

These expressions admit a Newtonian interpretation.

A Newtonian, viscous body deformed by rotation and tidal forces has a quadrupole moment given by

$$Q_{jk} = \frac{2}{3} n_2 R^5 \Omega_H^2 C_{jk} - \frac{2}{3} k_2 R^5 \left(\mathcal{E}_{jk} - \tau \dot{\mathcal{E}}_{jk} \right)$$

Love quantities

$$\tau \propto \nu R/M$$

in the body's rotating frame.

II. Tides: Newtonian body (2/3)

In the nonrotating frame this becomes

$$Q_{ab} = \frac{2}{3} n_2 R^5 \Omega_H^2 C_{ab} - \frac{2}{3} k_2 R^5 \left(\mathcal{E}_{ab} - \tau \dot{\mathcal{E}}_{ab} - \tau \Delta \dot{\mathcal{E}}_{ab} \right)$$

rotation-induced change in tidal moment

$$\Delta \dot{\mathcal{E}}_{ab} = 2 \Omega_H \epsilon_{cd(a} \mathcal{E}^c{}_{b)} s^d$$

II. Tides: Newtonian body (3/3)

The rate at which the tidal forces do work on the body is

$$\dot{W} = \frac{1}{3}(k_2\tau)R^5\,\dot{\mathcal{E}}^{ab}\big(\dot{\mathcal{E}}_{ab} + \Delta\dot{\mathcal{E}}_{ab}\big)$$

The rate at which they change its angular momentum is

$$\Omega_H\dot{J} = -\frac{1}{3}(k_2\tau)R^5\Delta\dot{\mathcal{E}}^{ab}\big(\dot{\mathcal{E}}_{ab} + \Delta\dot{\mathcal{E}}_{ab}\big)$$

The rate at which heat is dissipated by viscosity is

$$\dot{Q} = \dot{W} - \Omega_H\dot{J} = \frac{1}{3}(k_2\tau)R^5\,\big(\dot{\mathcal{E}}^{ab} + \Delta\dot{\mathcal{E}}^{ab}\big)\big(\dot{\mathcal{E}}_{ab} + \Delta\dot{\mathcal{E}}_{ab}\big)$$

II. Correspondence: nonrotating

The nonrotating black hole corresponds to

$$\dot{\mathcal{E}}_{ab} \gg \Delta\dot{\mathcal{E}}_{ab}$$

The Newtonian expressions reduce to

$$\dot{M} = \frac{1}{3}(k_2\tau)R^5\dot{\mathcal{E}}^{ab}\dot{\mathcal{E}}_{ab}, \qquad \dot{J} = -\frac{2}{3}(k_2\tau)R^5\left(\epsilon^a{}_{cd}\mathcal{E}^{cb}s^d\right)\dot{\mathcal{E}}_{ab}$$

This is to be compared with

$$\dot{M} = \frac{16}{45}M^6\dot{\mathcal{E}}^{ab}\dot{\mathcal{E}}_{ab}, \qquad \dot{J} = -\frac{32}{45}M^6\left(\epsilon^a{}_{cd}\mathcal{E}^{cb}s^d\right)\dot{\mathcal{E}}_{ab}$$

II. Correspondence: rotating

The rapidly-rotating black hole corresponds to

$$\dot{\mathcal{E}}_{ab} \ll \Delta\dot{\mathcal{E}}_{ab}$$

The Newtonian expressions reduce to

$$\dot{J} = -\frac{2}{3}(k_2\tau)R^5\Omega_H\Big[2\big(\mathcal{E}_{ab}\mathcal{E}^{ab}\big) - 3\big(\mathcal{E}_{ab}s^b\mathcal{E}^a{}_c s^c\big)\Big]$$

This is to be compared with

$$\sim (v_{\rm rot}/c)^2$$

$$\begin{aligned}\dot{J} \;=\; & -\frac{16}{45}M^6\Omega_H\Big(1+\sqrt{1-\chi^2}\Big)\Big[2(1+3\chi^2)\big(\mathcal{E}_{ab}\mathcal{E}^{ab}\big) \\ & - 3\Big(1+\frac{17}{4}\chi^2\Big)\big(\mathcal{E}_{ab}s^b\mathcal{E}^a{}_c s^c\big) + \frac{15}{4}\chi^2\big(\mathcal{E}_{ab}s^a s^b\big)^2\Big]\end{aligned}$$

II. Correspondence

In each case (nonrotating and rapidly-rotating) the correspondence produces agreement between all numerical factors, provided that

$$(k_2\tau)R^5 = \frac{16}{15}\left(\frac{GM}{c^3}\right)\left(\frac{GM}{c^2}\right)^5$$

This implies that

$$R \sim \frac{GM}{c^2}, \qquad k_2\tau \sim \frac{GM}{c^3} \neq 0$$

The horizon's effective viscosity [Hartle] is $k_2\nu \sim GM/c$

Conclusion

- ✓ Electric-type and magnetic-type tidal Love numbers can be defined unambiguously in general relativity; they have gauge-invariant significance.
- ✓ The tidal Love numbers of a nonrotating black hole are zero.
- ✓ The tidal interaction of black holes is deeply analogous to the tidal interaction of Newtonian, viscous bodies; the correspondence holds to near-quantitive accuracy.
- ✓ It implies that $k_2 \tau \neq 0$
  even though $k_2 = 0$

Epilogue

It is often said that for compact bodies, the effacement principle should break down at 5PN order:

$$a_j^{\rm tidal} \sim \frac{(GM)^4}{c^{10}} \mathcal{E}_{jkn} \mathcal{E}^{kn} \qquad \Rightarrow \qquad \frac{a^{\rm tidal}}{a^{\rm N}} \sim \frac{(GM)^5}{c^{10} r^5} \sim (v/c)^{10}$$

But another type of tidal coupling is possible (seen in the nonlinear theory of tidally-deformed black holes):

$$a_j^{\rm tidal} \sim \frac{(GM)^3}{c^9} \epsilon_{jkn} \mathcal{E}^k{}_p \mathcal{B}^{pn} \qquad \Rightarrow \qquad \frac{a^{\rm tidal}}{a^{\rm N}} \sim \frac{(GM)^4}{c^9 r^4} \sim (v/c)^9$$

Does the effacement principle break down at 4.5PN order?