Stochastic Inflation
and
Replica Field Theory

Florian Kühnel

University of Bielefeld

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with Dominik Schwarz

Overview

- Stochastic inflation
  - General idea
  - Statement of the problem

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  - Gaussian variational method
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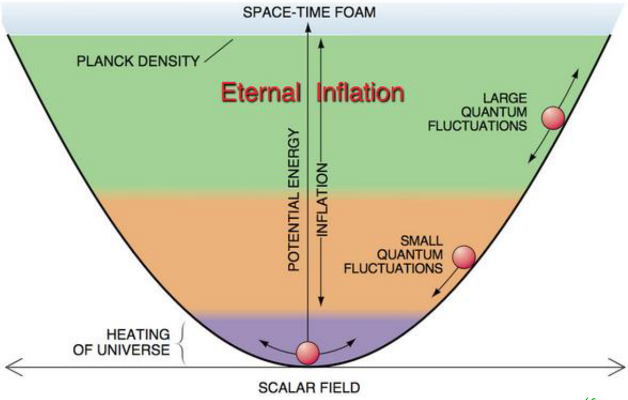
  - ... in the light of replica field theory
  - Results
- Lunch :-)

General Picture: What do we Describe?



(from Andrei Linde)

Stochastic Inflation I

Consider a free, minimally coupled, real scalar test field φ with mass μ, in d-dimensional space-time with metric $g_{\mu\nu}$.

Focus on de Sitter space-time:

$$(g_{\mu\nu}) = \mathrm{diag}\big(-1, a^2(t), \ldots, a^2(t)\big), \quad a(t) = \mathrm{e}^t, \quad H \stackrel{!}{=} 1$$

Split into long and short wavelengths:

$$\boxed{\varphi = \varphi_\mathrm{L} + \varphi_\mathrm{S}}$$

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$$\boxed{\varphi = \varphi_\mathsf{L} + \varphi_\mathsf{S}}$$

where

$$\varphi_\mathsf{S}(t, \mathbf{x}) = \int \mathrm{d}^{d-1}k \; \mathrm{W}_\kappa\left(\frac{k}{a(t)} - \epsilon\right)\left[\hat{\mathrm{a}}(k)\, u(t, k) + \mathrm{H.c.}\right]$$

with $\mathrm{W}_\kappa \xrightarrow{\kappa \to 0} \Theta$, where $0 < \epsilon \ll 1$.

Stochastic Inflation II

Think of φ_{S} as generating a (quantum) bath in which φ_{L} evolves.

[Starobinsky '85]

The field equation $(\Box + \mu^2)\,\varphi = 0$ implies

$$\boxed{(\Box + \mu^2)\varphi_{\mathsf{L}} = \mathrm{h}}$$

where h includes φ_{S} and is a Gaussian stochastic variable
$\rightsquigarrow$ Gaussian random force $\quad\leftrightarrow\quad$ random potential $\mathrm{V}_D(\varphi_{\mathsf{L}})$

This captures the leading-log$[a(t)]$ contribution to $\epsilon, p, \ldots$.

[Woodard, et al. '05]

So far so good...

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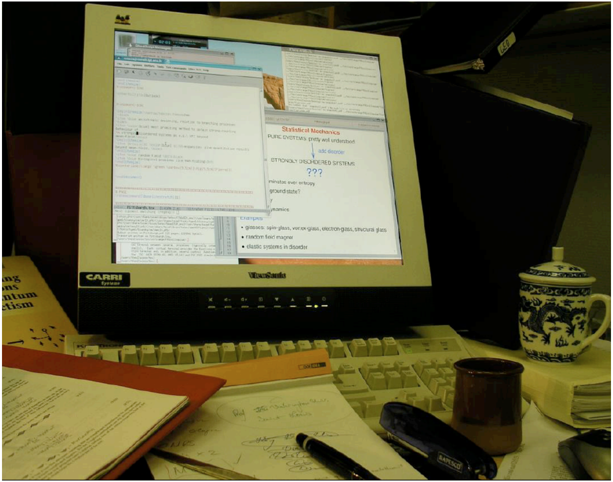
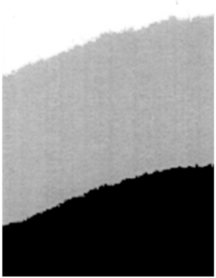
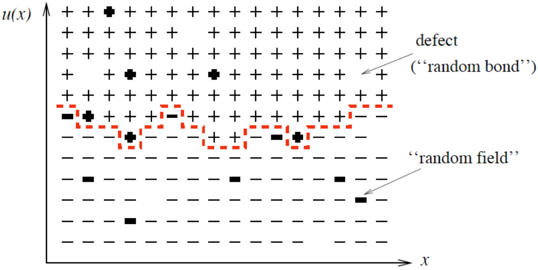
$\rightsquigarrow$ Use replica trick and Gaussian variational method! [F.K., Schwarz '08]

Disorder


Statistical Mechanics
PURE SYSTEMS: pretty well understood
add disorder
STRONGLY DISORDERED SYSTEMS
???
ground state?
glasses: spin-glass, vortex-glass, electron-glass, structural glass
random field magnet
elastic systems in disorder
CARRI
ViewSonic

New Methods... from Statistical Physics

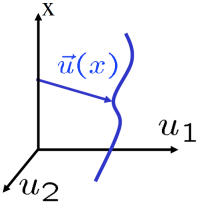


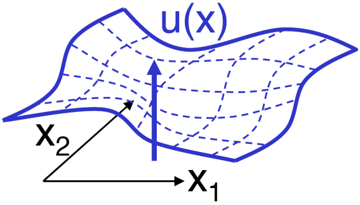


(from Kay Wiese)

Ingredients

field	$\vec{u}(x) \in \mathbb{R}^N$,	$x \in \mathbb{R}^d$	deformations
elastic energy	$\mathcal{H}_0[\vec{u}]$		free part
disorder energy	$\mathcal{H}_D[\vec{u}]$		impurities





The System

Describe: Gibbs equilibrium at temperature T

$$\mathcal{P}[\vec{u}] = \frac{1}{\mathcal{Z}} \exp\Big\{ -\beta\,\mathcal{H}[\vec{u}] \Big\}\,, \quad \beta := 1/T$$

with partition function

$$\mathcal{Z} = \int \mathcal{D}[\vec{u}]\, \mathrm{e}^{-\beta\,\mathcal{H}[\vec{u}]}$$

where (idealised)

$$\mathcal{H}[\vec{u}] := \mathcal{H}_0[\vec{u}] + \mathcal{H}_D[\vec{u}] = \frac{1}{2}\int_x \big(\nabla \vec{u}(x)\big)^2 + \int_x \mathrm{V}_D\big(x,\, \vec{u}(x)\big)$$

Replica Trick

Observables are averages.

Average $\overline{\mathcal{F}} = -T\,\overline{\ln\{\mathcal{Z}\}}$ is complicated.

$\rightsquigarrow$ Use replica trick

$$\overline{\mathcal{F}} = -T \lim_{n\to 0} \frac{1}{n} \ln\left\{\overline{\mathcal{Z}^n}\right\}$$

That means: compute

$$\overline{\mathcal{Z}^n} = \int \prod_{a=1}^{n} \mathcal{D}[\vec{u}_a]\, \overline{\exp\left\{-\beta \sum_{b=1}^{n} \mathcal{H}[\vec{u}_b]\right\}}$$

for $n \in \mathbb{N}$, and perform limit $n \to 0$ at the end.

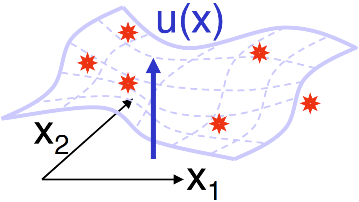
Cumulants

Suppose $\mathrm{V}_D\big(x,\, \vec{u}_a(x)\big)$ is Gaussian distributed with

$$\overline{\mathrm{V}_D\big(x,\, \vec{u}_a(x)\big)} = 0$$

$$\overline{\mathrm{V}_D\big(x,\, \vec{u}_a(x)\big)\, \mathrm{V}_D\big(y,\, \vec{u}_b(y)\big)} = \phi(x-y)\, \mathrm{R}\big(\vec{u}_a(x)\cdot \vec{u}_b(y)\big)$$

Special case I: $\phi(x-y) = \delta^{(d-1)}(x-y) \quad \rightsquigarrow \quad$ point-like disorder



(from Pierre Le Doussal)

Special case II: $\mathrm{V}_D\big(\vec{u}_a\big) \sim \vec{u}_a \quad \rightsquigarrow \quad \mathrm{R}\big(\vec{u}_a \cdot \vec{u}_b\big) \sim \vec{u}_a \cdot \vec{u}_b$

Gaussian Variational Method

Any trial Hamiltonian $\mathcal{H}_0 := \frac{1}{2}\sum_{ab}\int_k \mathrm{G^{-1}}_{ab}(k)\,\vec{u}_a(k)\cdot\vec{u}_b(-k)$ implies

$$\mathcal{F} \leq \mathcal{F}_0 + \left\langle \mathcal{H} - \mathcal{H}_0 \right\rangle_{\mathcal{H}_0} =: \mathcal{F}_{var} \quad \text{Feynman-Jensen}$$

with $\mathcal{F}_{(0)} := -T \, \ln \mathrm{Tr}\Big\{ \exp\big\{ -\beta\,\mathcal{H}_{(0)} \big\} \Big\}$

Make the ansatz

$$\mathrm{G^{-1}}_{ab}(k) := \mathrm{G}_0^{-1}(k)\,\delta_{ab} - \sigma_{ab}$$

Use self-energy-matrix (σ_{ab}) as a variational parameter.

One finds

$$\boxed{\sigma_{ab}(p) = \int_x \phi(x)\,\mathrm{e}^{\mathrm{i}px}\,\widehat{\mathrm{R}}'\left(\int_k \mathrm{e}^{\mathrm{i}kx}\mathrm{G}_{ab}(k)\right)}$$

Application to Stochastic Inflation

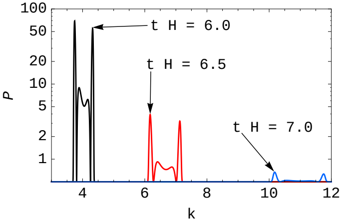
Presented methods imply

$$\boxed{\mathrm{G}(t,k) = \mathrm{G}_0(t,k) + \sigma(t,k)\mathrm{G}_0^2(t,k)}$$

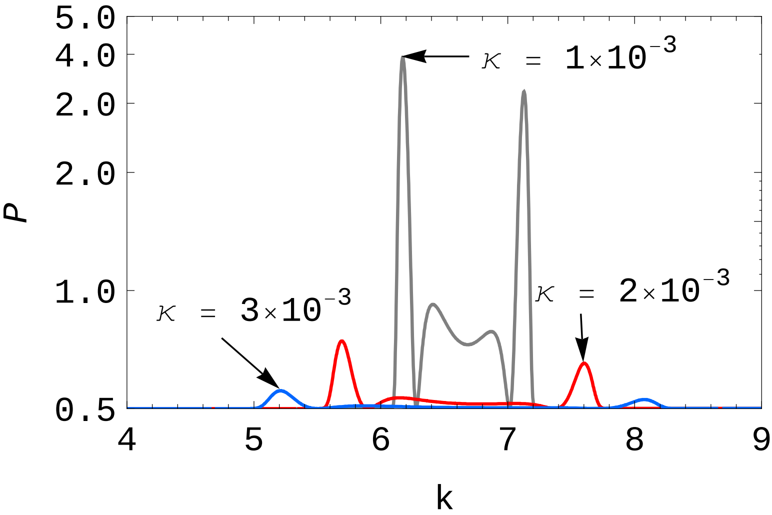
$\rightsquigarrow$ new dimensional reduction part $\leftrightarrow$ $\begin{cases} \text{modelling "non-Gaussianity"!} \\ \text{violation of scale invariance!} \end{cases}$

This implies for the dimensionless power spectrum

$$\mathcal{P}(k) \propto k^{d-1}\,\mathrm{G}(k)$$



Width of Filter Functions



Non-Linearity I

We had

$$\mathrm{G}(t,k) = \mathrm{G}_0(t,k) + \sigma(t,k)\,\mathrm{G}_0^2(t,k)$$

Consider general field

$$\varphi := \varphi_0 + g_{NL}\,\varphi_0^2$$

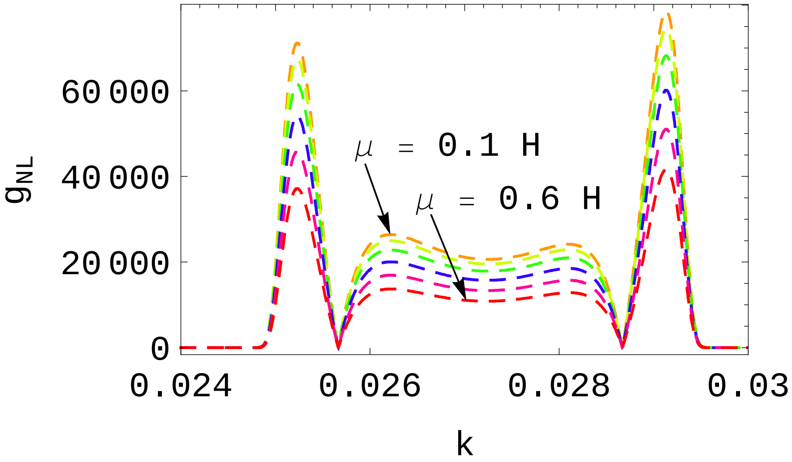
for a Gaussian field φ_0. Therefore:

$$\mathrm{G}(t,k) = \mathrm{G}_0(t,k) + 3\,g_{NL}^2\,\mathrm{G}_0^2(t,k)$$

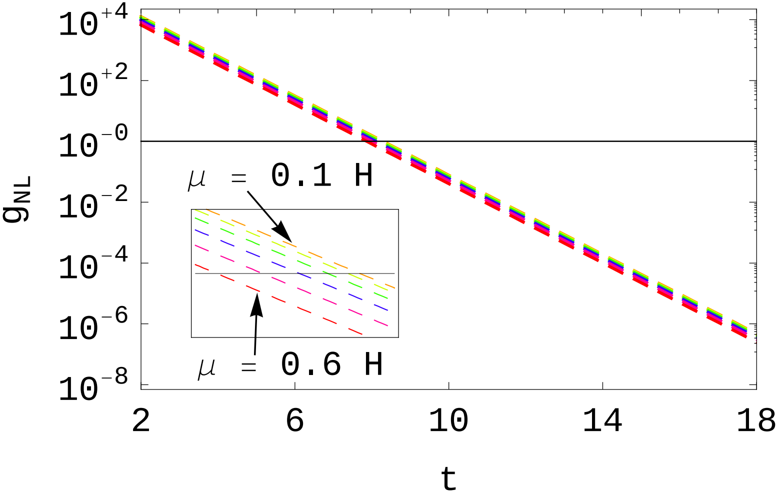
and thus

$$\boxed{g_{NL}^2(t,k) = \frac{1}{3}\,\sigma(t,k)}$$

Non-Linearity II



Non-Linearity III



Angular Two-Point Function

Two-point correlation of observed CMB temperature fluctuations

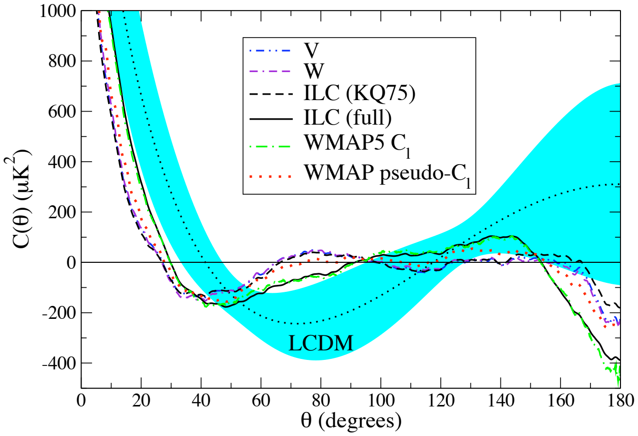
$$\mathcal{C}(\theta) := \left\langle \Delta \mathrm{T}(\hat{\mathbf{e}}_1)\, \Delta \mathrm{T}(\hat{\mathbf{e}}_2) \right\rangle_\theta = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1)\, C_\ell\, \mathrm{P}_\ell(\cos\theta)$$

Angle brackets represent average over all pairs of points on the sky.

C_ℓ is given through

$$C_\ell = \frac{4\pi}{9} \int_0^\infty \frac{\mathrm{d}k}{k} \left[\mathrm{j}_\ell\left(\frac{2k}{H_0} \right) \right]^2 \mathcal{P}(k)$$

No Large-Angle Correlation

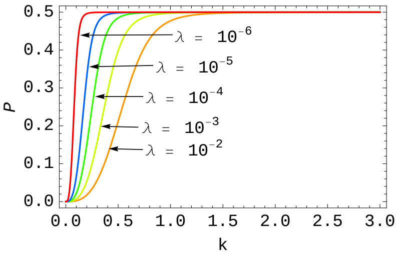


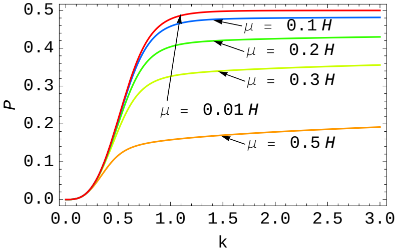
Include now interactions,
specifically a φ^4 self-coupling:

$$\sigma_{ab}(k) = \phi(k) - 8\,\lambda\,\mathrm{G}_{aa}(k)\,\delta_{ab}$$

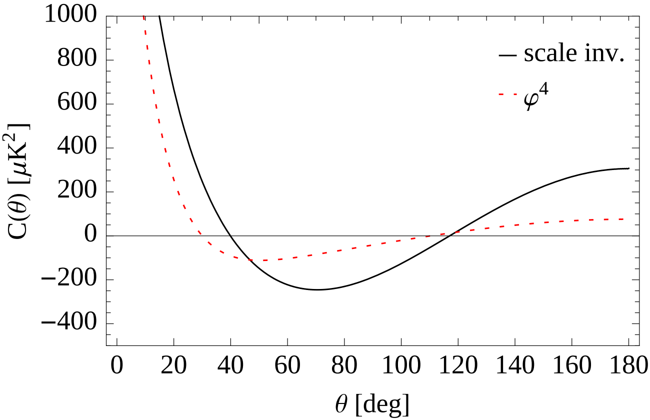
From which we determine $\mathcal{P}(k)$

$$\mathcal{P}(k) = \frac{k^3}{\mathrm{G}_0^{-1}(k) + 8\,\lambda\,k^{-3}\mathcal{P}(k)} + \frac{k^3\,\phi(k)}{\left[\mathrm{G}_0^{-1}(k) + 8\,\lambda\,k^{-3}\mathcal{P}(k)\right]^2}$$





Comparison



1000
800
600
400
200
0
−200
−400
C(θ) [μK²]
— scale inv.
φ⁴
0 20 40 60 80 100 120 140 160 180
θ [deg]

Summary and Outlook

- In this talk
  - Application of replica field theory to stochastic inflation
  - Cosmological dimensional reduction
    $\longrightarrow$ deviation from scale invariance
  - Damping on large scales
  - Explain absence of large-angle correlation
- Currently working on
  - Low-multipole suppression
  - Arbitrary potentials $\leftrightarrow$ replica symmetry breaking [Bray, ...]
  - Relation to functional renormalisation group [Le Doussal, ...]
- Open problem: Stochastic geometry