# Critical Phenomena in Gravitational Collapse

José M. Martín-García

Laboratoire Univers et Theories
&
Institut d'Astrophysique de Paris

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# Summary

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1. Historical introduction
   - Christodoulou
   - Choptuik
   - Other results
2. The current model
   - Self-similarity
   - GR as a dynamical system
   - Mass scaling
3. Interesting results
   - Non-spherical systems
   - Global structure of the critical solution
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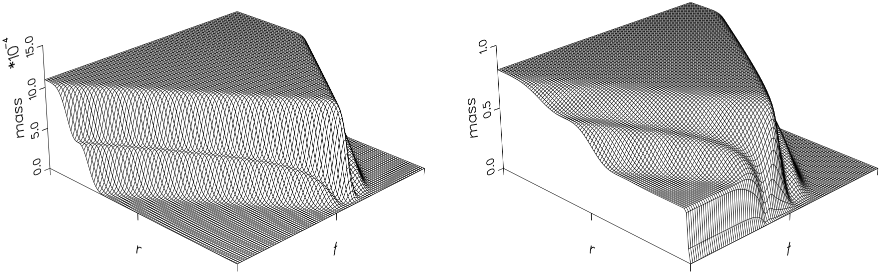

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- Goldwirth and Piran, PRD'87:

  *We present a numerical study of the gravitational collapse of a massless scalar field. We calculate the future evolution of new initial data, suggested by Christodoulou, and we show that in spite of the original expectations these data lead only to singularities engulfed by an event horizon.*

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- And then in 1993...

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- The system:

$$ds^2 = -\alpha^2(t,r)dt^2 + a^2(t,r)dr^2 + r^2 d\Omega^2, \qquad \Phi \equiv \phi', \quad \Pi \equiv a\dot{\phi}/\alpha$$

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)', \qquad \dot{\Pi} = \frac{1}{r^2}\left(r^2\frac{\alpha}{a}\Phi\right)', \qquad \frac{\alpha'}{\alpha} = \frac{a'}{a} + \frac{a^2-1}{r} = 2\pi r(\Pi^2 + \Phi^2).$$

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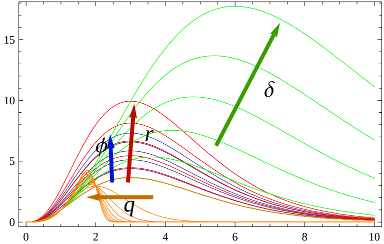
Example (pure ingoing):

$$\phi(0,r) = \phi_0\, r^3\, \exp\left(-[(r-r_0)/\delta]^q\right)$$

$$p = \phi_0, r_0, \delta, q$$



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Comment: Self-similarity is dynamically found, but in a more general form!

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Phenomenology confirmed in more than 20 other systems:

- Abrahams & Evans (PRL'93): axisymmetric vacuum (DSS).
- Evans & Coleman (PRL'94): perfect fluid, $p = \rho/3$ (CSS).
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- Proca, Dirac, sigma fields, ..., Vlasov(?)
- With/without mass, charge, conformal couplings, ...
- Different equations of state for fluids.
- Other dimensions.

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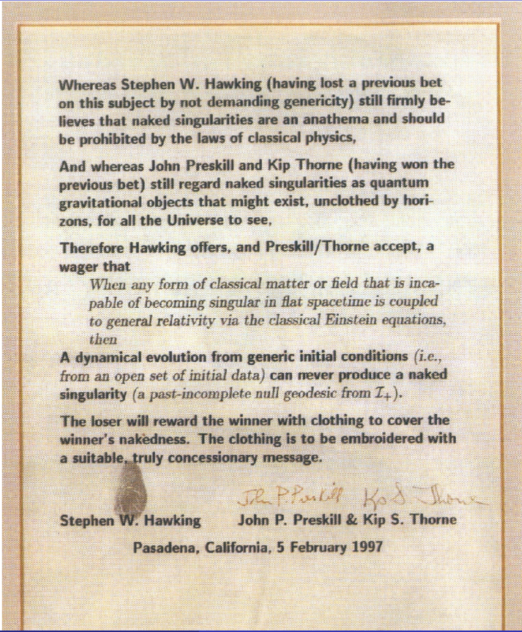

Whereas Stephen W. Hawking (having lost a previous bet on this subject by not demanding genericity) still firmly believes that naked singularities are an anathema and should be prohibited by the laws of classical physics,

And whereas John Preskill and Kip Thorne (having won the previous bet) still regard naked singularities as quantum gravitational objects that might exist, unclothed by horizons, for all the Universe to see,

Therefore Hawking offers, and Preskill/Thorne accept, a wager that

> *When any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, then*

A dynamical evolution from generic initial conditions *(i.e., from an open set of initial data)* can never produce a naked singularity *(a past-incomplete null geodesic from $\mathcal{I}_+$).*

The loser will reward the winner with clothing to cover the winner's nakedness. The clothing is to be embroidered with a suitable, truly concessionary message.

Stephen W. Hawking — John P. Preskill & Kip S. Thorne

Pasadena, California, 5 February 1997

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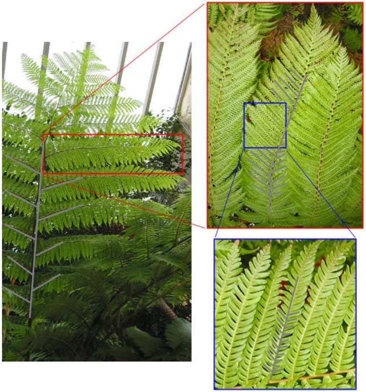
- Any continuous symmetry has a discrete version.



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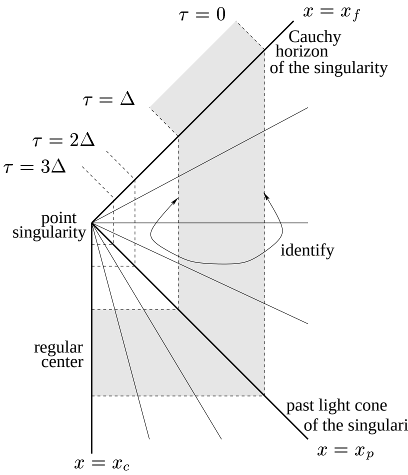

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- Then any metric is:

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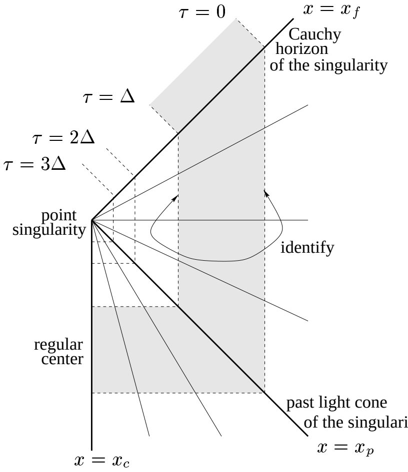
  with $\xi = \partial_\tau$.



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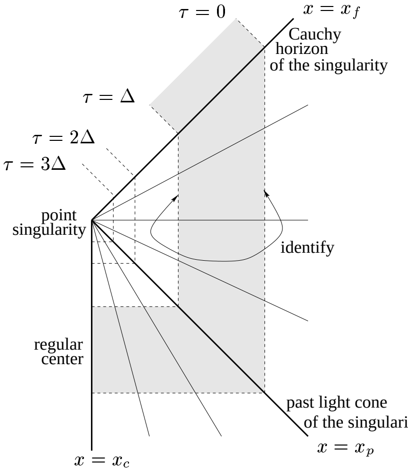
- CSS: $A, B, C, F$ functions of $x$ only. DSS: also periodic in $\tau$, period $\Delta$.



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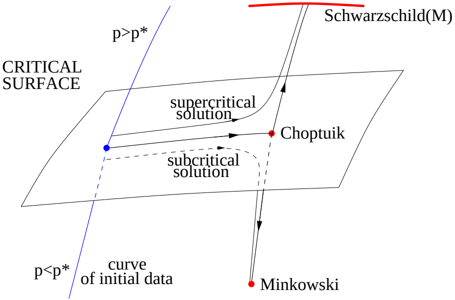
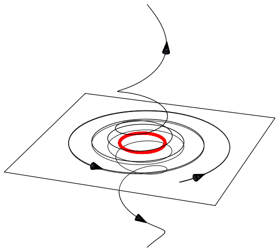
- Evolution in ($\infty$-dim) phase space:



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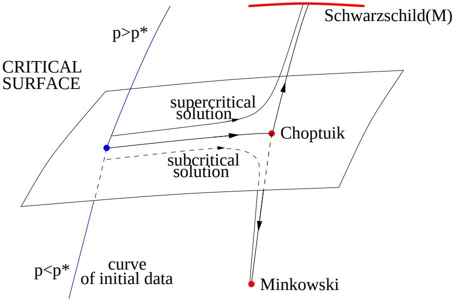


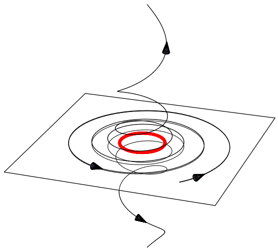

- Warnings:
  - Which functional space? Asymptotic properties of the spacetimes.
  - Which foliations? Which coordinates?
  - Meaning of "attraction"?

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  - Static ("type I") or self-similar ("type II"). Both continuous or discrete.
- Two approaches to criticality:
  - The nonlinear way:
    - Evolution code and fine tune IC families to critical surface.
    - Search for critical phenomena (mainly universality).
  - The linear way:
    - Construct candidate critical solution.
    - Check there is a unique unstable linear mode.

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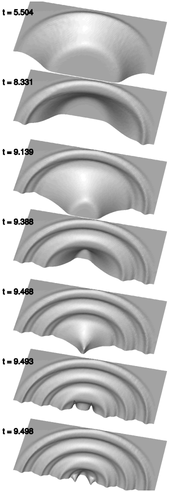

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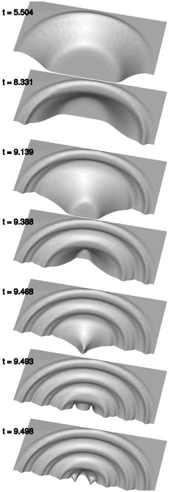

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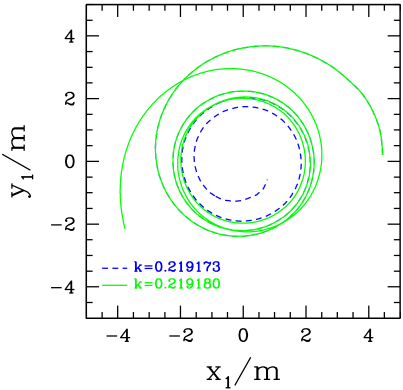

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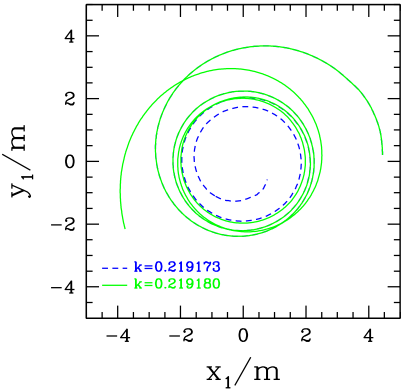

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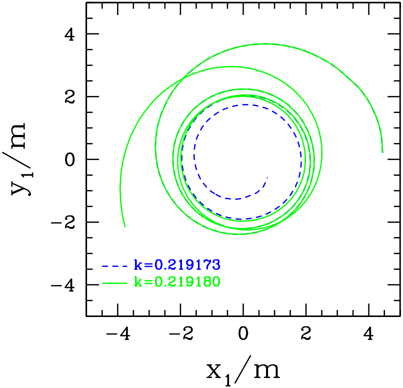

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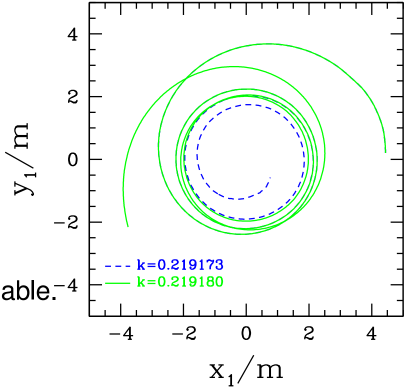

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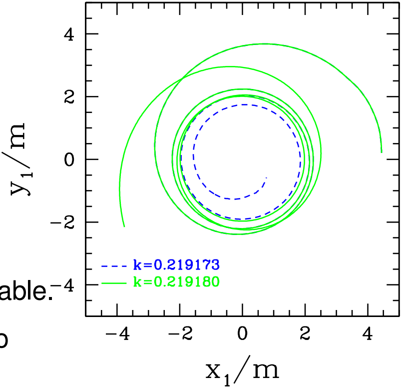

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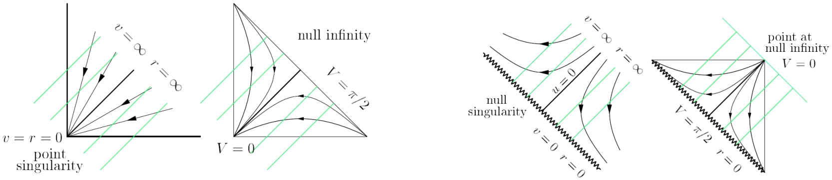

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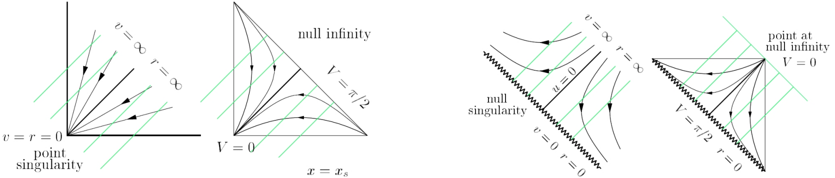
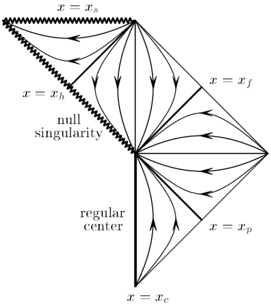



- Example:



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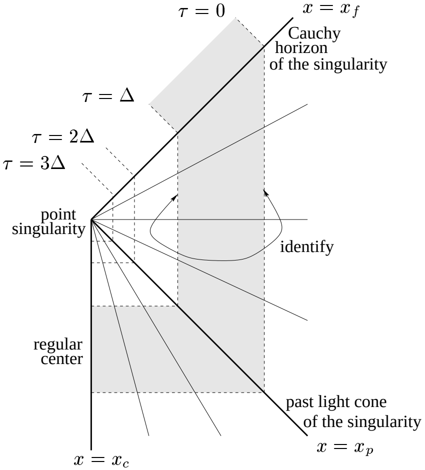
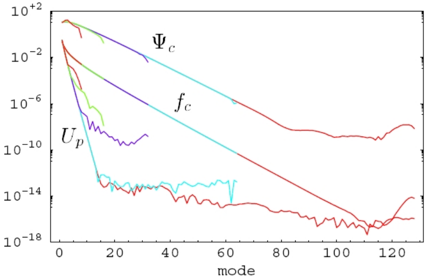
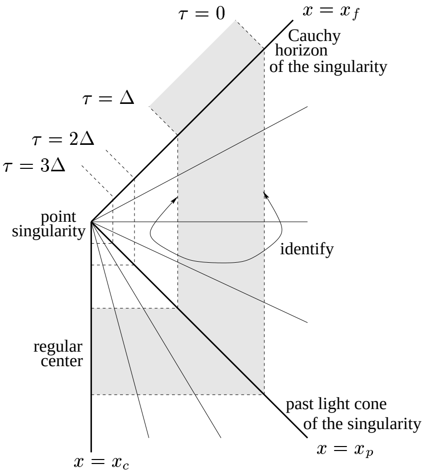


JMMG & Gundlach PRD'03

- Three regions
- Psedospectral code. Fourier in $\tau$; $4^{th}$ order FD in *x*.



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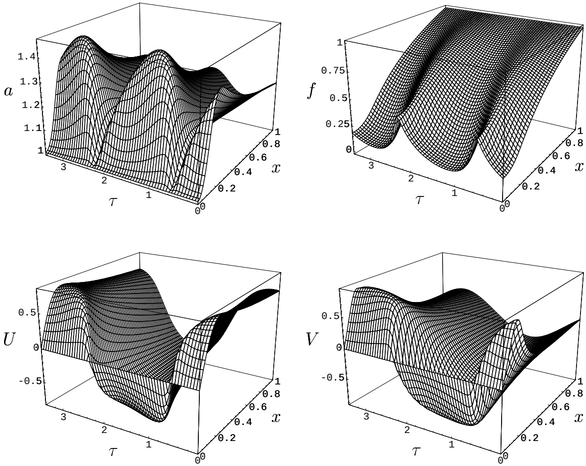

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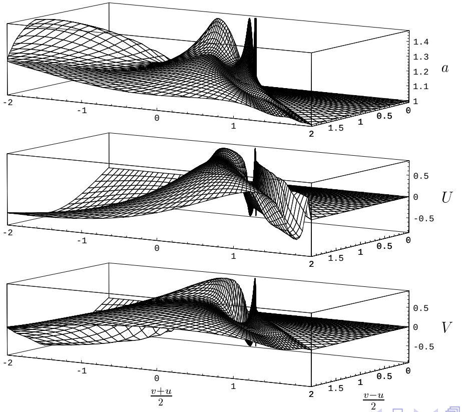
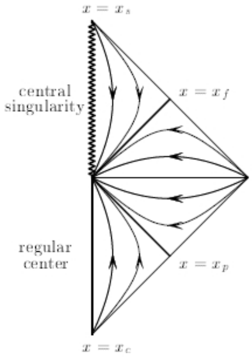
- Curvature is continuous but non-differentiable. Continuation not unique: one free function (radiation from the singularity).
- Unique DSS continuation with regular center (nearly flat):



# 3.2. Global structure of the Choptuik spacetime

All other continuations produce a negative mass singularity at the centre, with no new self-similarity horizon:



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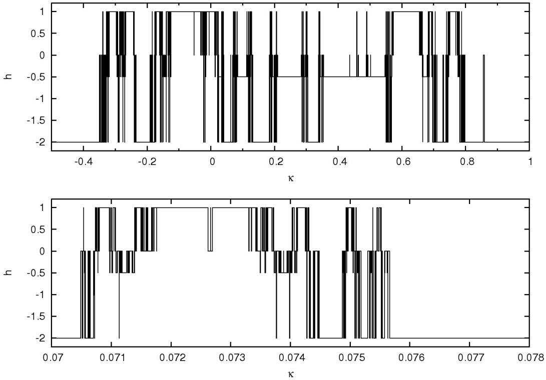

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- Quantum and astrophysical relevance?

# Conclusions and open questions

- It is “easy” to form a naked singularity: fine tune to the BH threshold.
- Process controlled by an exact solution (“choptuon”), as gravitational collapse is controlled by a black hole.
- Route to visible regions with arbitrarily high curvature.
- First qualitative pictures of GR phase space. Chaos.
- Numerical Relativity can add new physics to mainstream GR. Note the importance of very high precision numerics.

- Dynamical understanding of the process missing.
- What happens outside spherical symmetries? No 3D simulations so far.
- Show existence of the Choptuik spacetime.
- Can we approximate critical exponents analytically? Holography?
- Quantum and astrophysical relevance?

Gundlach & JMMG, Living Reviews Relativity 2007.