Dynamics of super-inflation in Loop Quantum Cosmology

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- **Slow-roll from LQC $k = 0$ and $k = 1$**
- **Super-inflation in LQC**
- **Number of e-folds**

J. E. Lidsey, D. J. Mulryne, NJN, R. Tavakol (2004)
D. J. Mulryne, NJN, R.Tavakol, J. E. Lidsey (2004)
Mulryne, Nunes (2006)
Copeland, Mulryne, Nunes, Shaeri (2007)

1. Loop Quantum Gravity

Theory of Gravity based on Ashtekar's variables which brings GR into the form of a gauge theory.

- Densitized triad E_i^a and $E_i^a E_i^b = q^{ab} q$
- SU(2) connection $A_a^i = \Gamma_a^i - \gamma K_a^i$

Γ_a^i - spin connection; K_a^i - extrinsic curvature; γ - Barbero-Immirzi parameter.

Quantization proceeds by using as basic variables holonomies,

$$h_e = \exp \int_e \tau_i A_a^i \dot{e}^a dt$$

in edges e, and fluxes,

$$F = \int_S \tau^i E_i^a n_a d^2 y$$

in spacial surfaces S.

2. Loop Quantum Cosmology

Focuses on minisuperspace settings with finite degrees of freedom.

Evolution of the Universe can be divided into 3 distinct phases:

- Quantum phase: $a < a_i$ and $a_i^2 = \gamma \ell_{\rm pl}^2$.

Described by a difference equation;

- Semi-classical phase: $a_i < a < a_*$.

$$a_*^2 = \frac{j}{3} a_i^2$$

Continuous evolution but equations modified due to non-perturbative quantization effects;

- Classical phase: $a > a_*$.

Usual continuous cosmological equations.

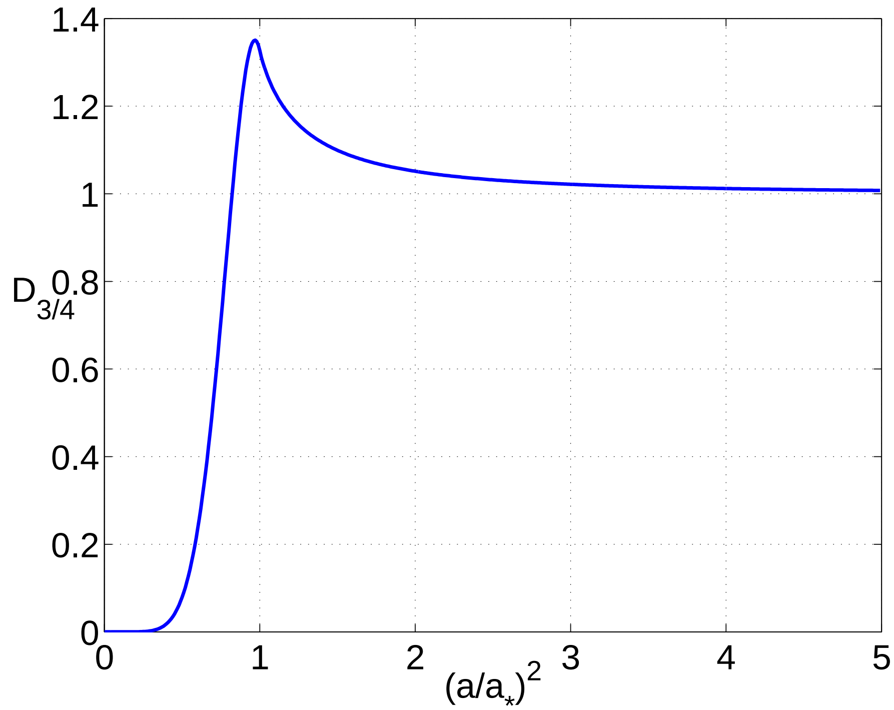
3. Inverse volume operator

Classically: $d(a) = a^{-3}$

LQC: $d_{l,j}(a) = D_l(q) a^{-3}$ where $q = \left(\frac{a}{a_*}\right)^2$

for $a \ll a_*$, $D(q) \approx D_\star a^n$

for $a \gg a_*$, $D(q) \approx 1$



4. Modified semi-classical equations

1. Modified Friedmann equation

   Hamiltonian density is

$$\mathcal{H}_\phi = \frac{1}{2} d_{l,j}(a) p_\phi^2 + a^3 V(\phi)$$

   and from the Hamiltonian constraint $\mathcal{H} = 0$,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{S}{3}\left(\frac{1}{2}\frac{\dot{\phi}^2}{D} + V(\phi)\right) - \frac{S^2}{a^2}$$

2. Modified Klein-Gordon equation

   From the Hamilton's equations

$$\dot{\phi} = \{\phi, \mathcal{H}_\phi\} = d_{l,j} p_\phi, \quad \dot{p}_\phi = \{p_\phi, \mathcal{H}_\phi\} = -a^3 \frac{dV}{d\phi}$$

   to obtain

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\left(1 - \frac{1}{3}\frac{d \ln D}{d \ln a}\right)\dot{\phi} + D\frac{dV}{d\phi} = 0$$

   Antifrictional term when $d \ln D / d \ln a > 3$ in expanding Universe and frictional term in a contracting Universe.

3. Variation of the Hubble rate

$$\dot{H} = -\frac{S\dot{\phi}^2}{2D}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a} - \frac{1}{6}\frac{d\ln S}{d\ln a}\right) + \frac{S}{6}\frac{d\ln S}{d\ln a}V$$

$$+\left(1 - \frac{d\ln S}{d\ln a}\right) S^2 \frac{1}{a^2}$$

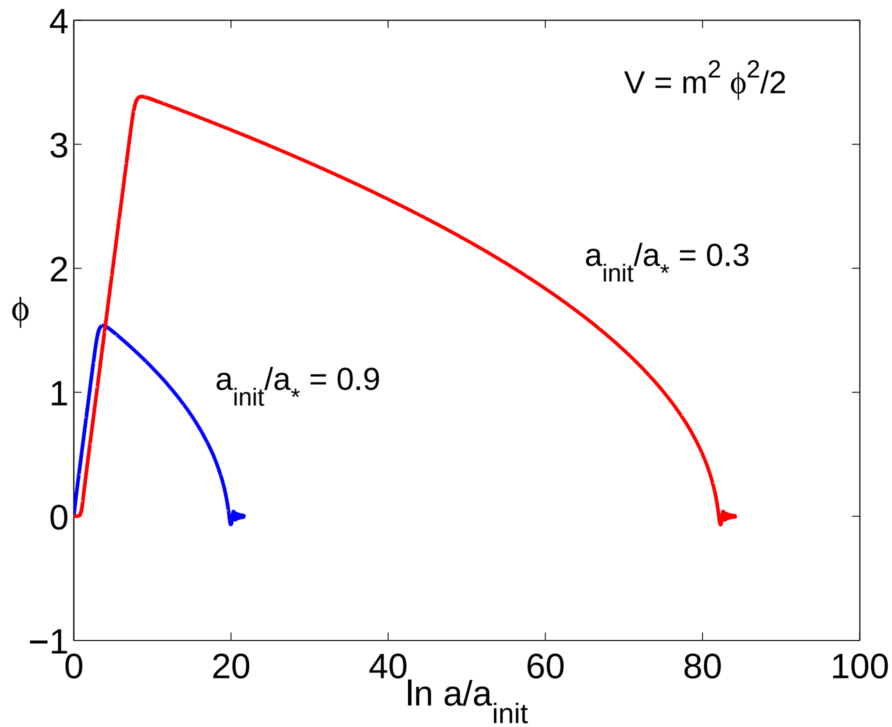
Super-inflation for $n + r = d\ln D/d\ln a + d\ln S/d\ln a > 6$.

4. Effective equation of state

$$w_{\text{eff}} = -1 + \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2DV}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a} - \frac{1}{6}\frac{d\ln S}{d\ln a}\right) - \frac{2}{3}\frac{DV}{\dot{\phi}^2 + 2DV}\frac{d\ln S}{d\ln a}$$

Super-inflation $(w < -1)$ when $n + r = d\ln D/d\ln a + d\ln S/d\ln a > 6$.

5. Consequences for inflation (flat Universe)



Tsujikawa and Singh (2003)

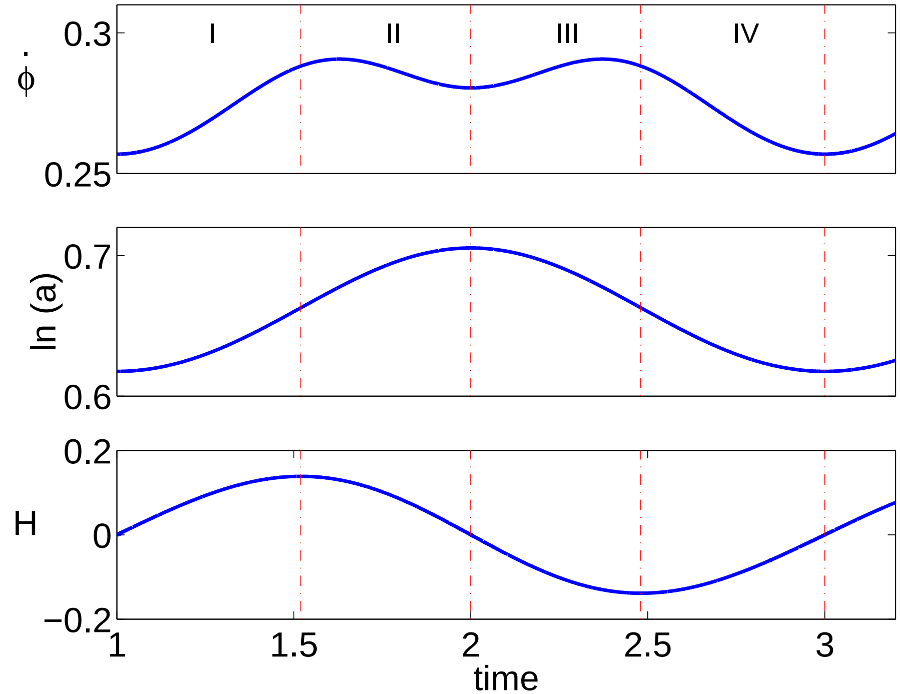
1. Super-inflation is brief;
2. ϕ_t independent of j;
3. $\phi_t \propto q_{\rm init}^{-6} \exp(-q_{\rm init}^{15/4})$;
4. $\phi_t < 2.4\ell_{\rm pl}^{-1}$ if Hubble bound ($1/H > a_i$) is satisfied $\Rightarrow$ not enough slow-roll inflation!

6. Bouncing Universe in $k = +1$, massless case

$$H^2 = \frac{1}{3}\left(\frac{\dot{\phi}^2}{2D} + V\right) - \frac{1}{a^2}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\left(1 - \frac{1}{3}\frac{d\ln D}{d\ln a}\right)\dot{\phi} + D\frac{dV}{d\phi} = 0$$

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{D}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right) + \frac{1}{a^2}$$



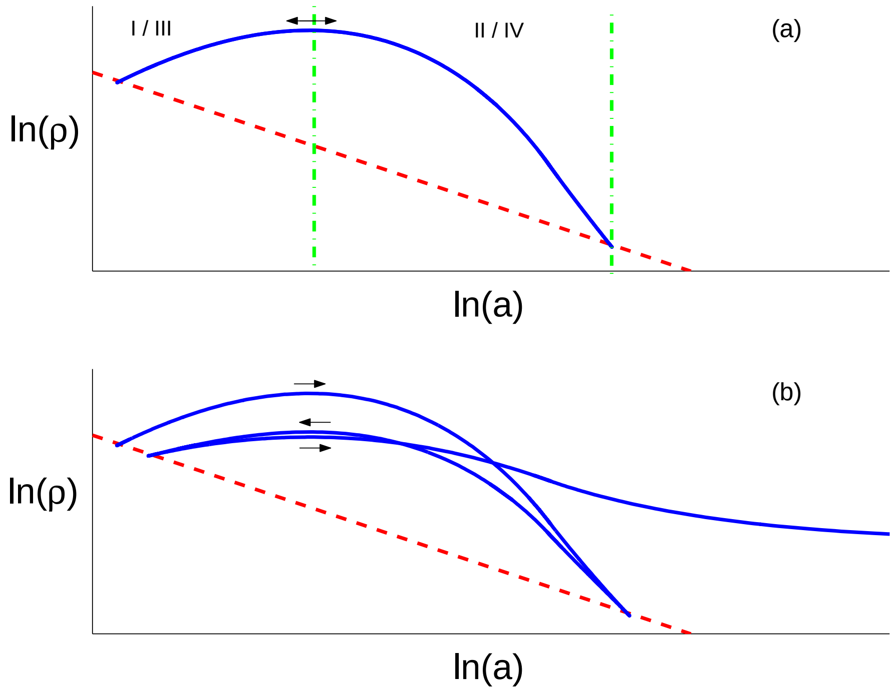
I - $a < a_*$, $a \nearrow$, $\dot{\phi} \nearrow$, superinflation

II - $a > a_*$, $a \nearrow$, $\dot{\phi} \searrow$, $H \searrow 0$, bounce

III - $a > a_*$, $a \searrow$, $\dot{\phi} \nearrow$, super-deflation

IV - $a < a_*$, $a \searrow$, $\dot{\phi} \searrow$, $H \nearrow 0$, bounce

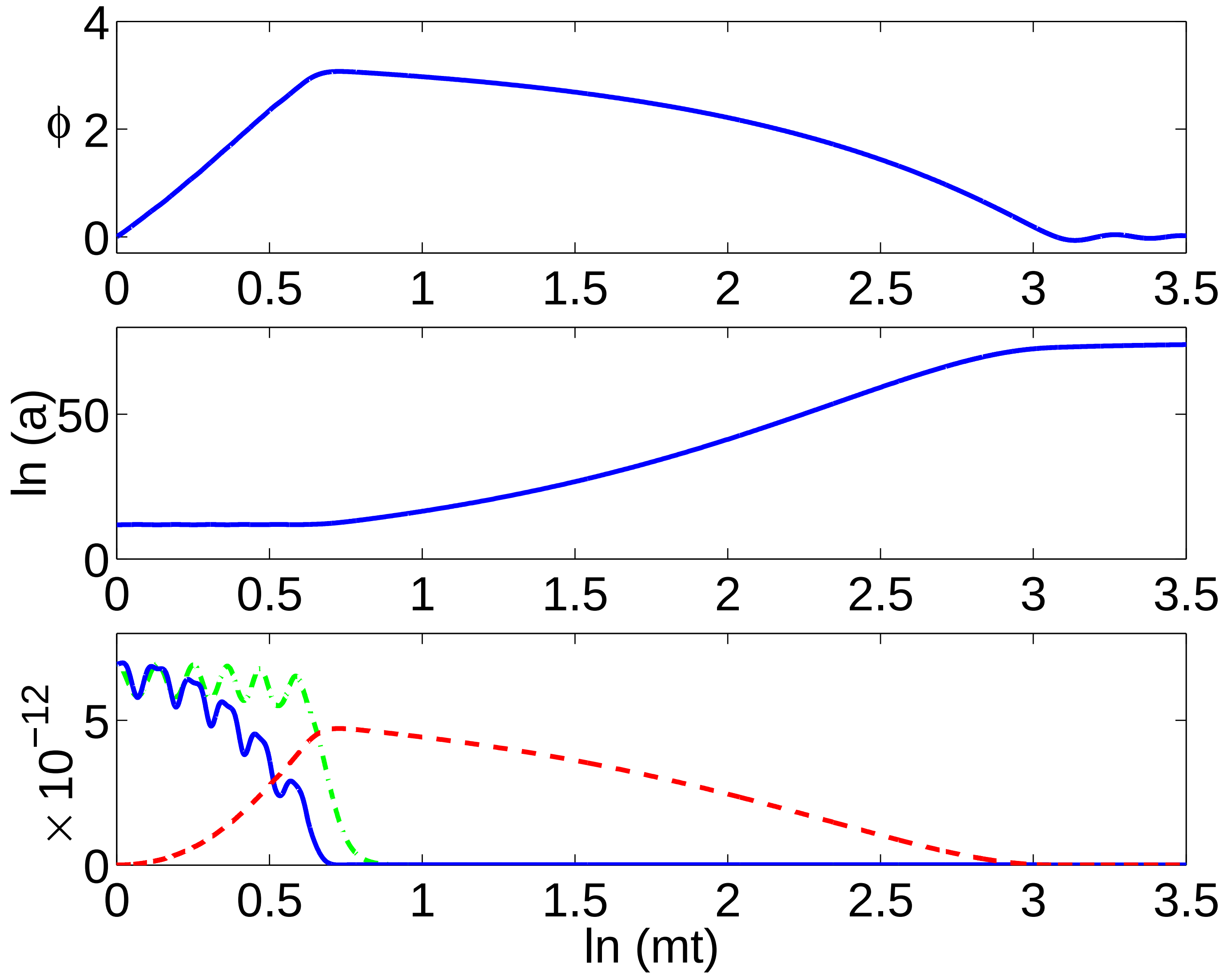
7. Bouncing Universe in $k = +1$, with self interacting potential



Field redshifts more rapidly than curvature term provided $\dot{\phi}^2 > V$ ($w > -1/3$).

As the field moves up the potential this condition becomes more difficult to satisfy and is eventually broken. Slow-roll inflation follows.

7. Bouncing Universe in $k = +1$, with self interacting potential



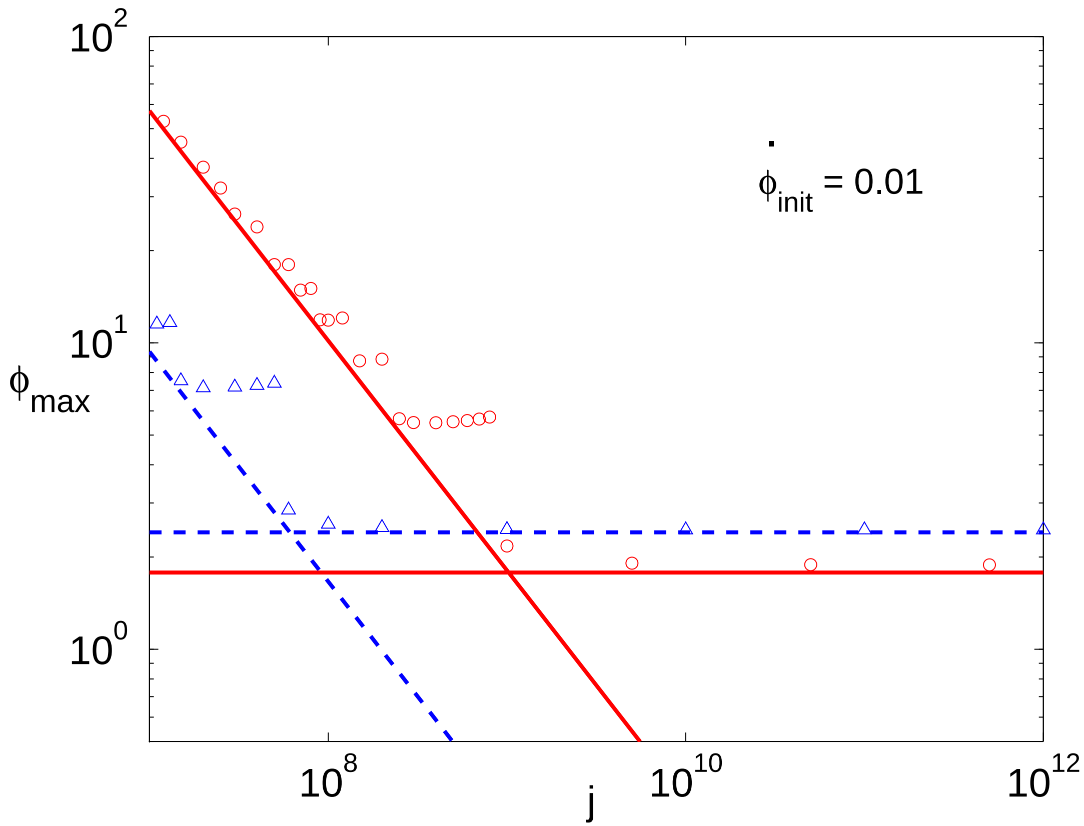
8. Value of the field at turn around, $\phi_{\rm t}$

1. $k = 0$

$$\phi_t \exp\left[\sqrt{12\pi}\ell_{\rm pl}(\phi_t - \phi_{\rm init})\right] = \frac{\dot\phi_{\rm init}}{m}\frac{\sqrt{2}}{q^6_{\rm init}}\left(\frac{7}{12}\right)^{24/5} \exp\left[\frac{6}{15}\left(1-(12/7)^3 q_{\rm init}^{15/4}\right)\right]$$

2. $k = 0$

$$\phi_t^2 = \frac{1}{\dot\phi_{\rm init}}\frac{2}{m}\left(\frac{8\pi\ell_{\rm pl}^2}{2}\right)^{-3/2}\frac{D_{\rm init}}{q_{\rm init}^{3/2}}\frac{1}{a_*^3}$$



9. The story so far...

1. Flat geometry
   - ϕ does not move high enough;
   - ϕ_t independent of quantization parameter j.

2. Positively curved geometry
   - Allows oscillatory Universe;
   - For massless scalar field cycles are symmetric and consequently ever lasting;
   - Presence of a self interaction potential breaks symmetry and establishes initial conditions for inflation;
   - *Low* j results into *more* inflation;
   - For sufficiently low j one can enter eternal inflation;

Can super-inflation during the semi-classical phase replace slow-roll inflation?

10. Scaling solution

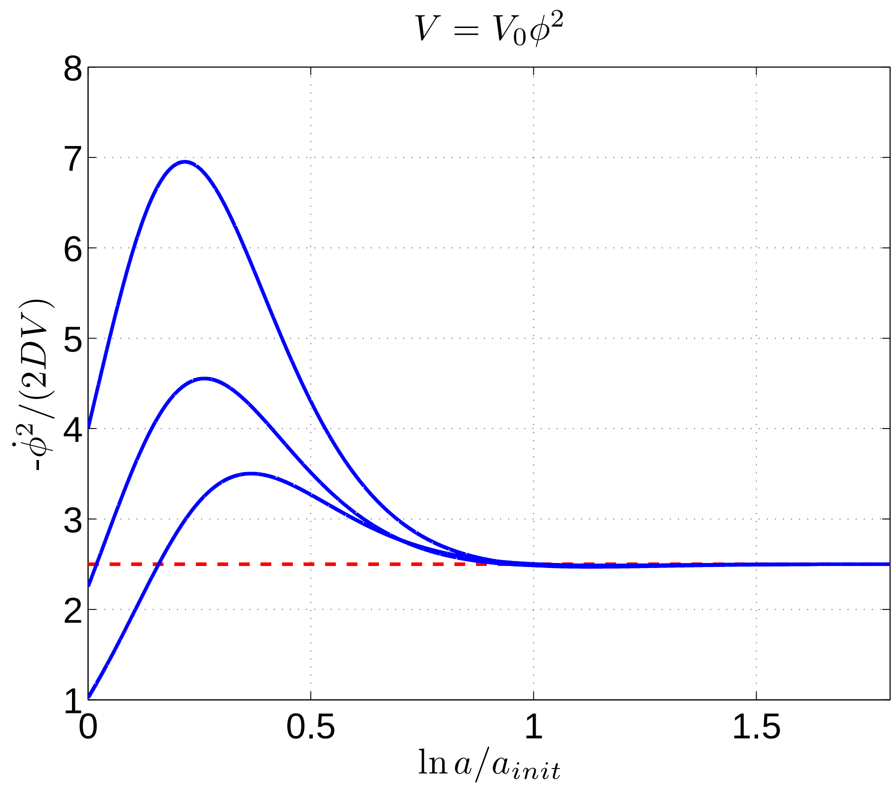
Scaling solution $\Leftrightarrow$ $\dot{\phi}^2/(2DV) \approx$ cnst.
Lidsey (2004)

$$a = (-\tau)^p$$

$$p = \frac{2\alpha}{2\bar{\epsilon} - (2+r)\alpha}$$

$$\bar{\epsilon} = \frac{1}{2}\frac{D}{S}\left(\frac{V_{,\phi}}{V}\right)^2$$

$$V = V_0\,\phi^\beta$$



$\beta = 4\bar{\epsilon}/(n-r)\alpha > 0$, $\alpha = 1 - n/6$, $D \propto a^n$, $S \propto a^r$.

Scaling solution is *stable* attractor for $\bar{\epsilon} > 3\alpha^2$ or $\beta > (n-6)/n \sim \mathcal{O}(1)$.

11. Perturbation equations

Define effective action that gives background equations of motion

$$S = \int d\tau\, d^3x\, a^4 \left(\frac{\phi'^2}{2Da^2} - \frac{\delta^{ij}}{a^2} \partial_i \phi \partial_j \phi - V \right)$$

Perturb field $\phi = \phi_b + \delta\phi$

Define $u = a\delta\phi/\sqrt{D}$ and get

$$\delta S = \int d\tau\, d^3x\, \left(u'^2 - D\delta^{ij} \partial_i u \partial_j u - m_{\rm eff}^2 u^2 \right)$$

$$m_{\rm eff}^2 = -\frac{(a/\sqrt{D})''}{a/\sqrt{D}} + a^2\, D\, \frac{\partial^2 V}{\partial \phi^2}$$

Action is $similar$ ⚠ to the action of a scalar field u in flat spacetime with time dependent effective mass. Can construct quantum theory in an analogous way to that of scalar field u propagating on Minkowski spacetime in the presence of background field ϕ .

12. Quantization

Momentum canonically conjugate to u

$$\pi(\tau, \mathrm{x}) = \frac{\partial \mathcal{L}}{\partial u'} = u'$$

Promote u and π to operators $\hat{u}$ and $\hat{\pi}$ s.t.

$$[\hat{u}(\tau, \mathrm{x}), \hat{u}(\tau, \mathrm{y})] = [\hat{\pi}(\tau, \mathrm{x}), \hat{\pi}(\tau, \mathrm{y})] = 0 \ , \qquad [\hat{u}(\tau, \mathrm{x}), \hat{\pi}(\tau, \mathrm{y})] = i\delta^{(3)}(\mathrm{x} - \mathrm{y})$$

Expand $\hat{u}$ in terms of plane waves:

$$\hat{u}(\tau, \mathrm{x}) = \int \frac{d^3\mathrm{k}}{(2\pi)^{3/2}} \left[\omega_k(\tau)\hat{a}_{\mathrm{k}} + \omega_k^*(\tau)\hat{a}^\dagger_{-\mathrm{k}} \right] e^{-i\mathrm{k}.\mathrm{x}}$$

Obtain equation of motion

$$\omega_k'' + \left(Dk^2 + m_{\mathrm{eff}}^2 \right) \omega_k = 0$$

$$\omega_k'' + \left(D_* A^n (-\tau)^{np} k^2 + \frac{m_{\mathrm{eff}}^2 \tau^2}{\tau^2} \right) \omega_k = 0$$

13. General solution

$$\omega_k'' + \left(D_* A^n (-\tau)^{np} k^2 + \frac{m_{\rm eff}^2 \tau^2}{\tau^2} \right) \omega_k = 0$$

Scaling solution $m_{\rm eff}^2 \, \tau^2 = -2 + (3-2n)p + \frac{1}{2}(6 - 2n - n^2)p^2$

General solution is:

$$\omega_k(\tau) = c_1 \sqrt{-\tau} J_{|\nu|}(x) + c_2 \sqrt{-\tau} Y_{|\ nu|}(x)$$

$$x \propto k(-\tau)^{(2+np)/2} \propto \frac{\sqrt{D}k}{aH}\,, \qquad \nu = -\frac{\sqrt{1-4m_{\rm eff}^2}}{2+np}$$

We are interested in the asymptotic behavior when $x \gg 1$ and $x \ll 1$. Define, by analogy with standard inflation, effective horizon $d_H = \frac{\sqrt{D}}{aH}$ black or effective wavenumber $k_* = \sqrt{D}k$.

14. Normalization and asymptotic limits

Ensure that creation and annihilation operators $\hat{a}^\dagger_{\rm k}$ and $\hat{a}_{\rm k}$ satisfy usual commutation relations $[\hat{a}_{\rm k}, \hat{a}_{\rm l}] = [\hat{a}^\dagger_{\rm k}, \hat{a}^\dagger_{\rm l}] = 0$, $[\hat{a}_{\rm k}, \hat{a}^\dagger_{\rm l}] = \delta^{(3)}({\rm k} - {\rm l})$

Using the commutation relations for u and π get

$$\omega^*_k \, \omega'_k - \omega_k \, \omega^{*\prime}_k = 0 \qquad \text{(Wronskian condition)}$$

General normalized solution is

$$\omega_k(\tau) = \sqrt{\frac{\pi}{2|2+np|}}\,\sqrt{-\tau}\, H^{(1)}_{|\nu|}(x)$$

1. Small wavelength limit ($\sqrt{D}k/aH \gg 1$)

$$H^{(1)}_\nu(x) \to \sqrt{\frac{2}{\pi x}}\, e^{i(x - \pi\nu/2 - \pi/4)}$$

$$\omega_k(\tau) = \frac{(-\tau)^{-np/4}}{\sqrt{|2+np|\alpha k}}\, e^{i\alpha k(-\tau)^{(2+np)/2}}, \qquad \alpha = 2\,\frac{\sqrt{D_*}}{|2+np|}$$

We *do not* recover flat spacetime solution ($\omega_k = e^{-ik\tau}/\sqrt{2k}$) unless $n = 0$ (which only happens at the end of the superinflationary phase).

15. Normalization and asymptotic limits (*cont.*)

2. Long wavelength limit ($\sqrt{D}k/aH \ll 1$)

$$J_{|\nu|}(x) \to \frac{1}{\Gamma(|\nu|+1)} \left(\frac{x}{2}\right)^{|\nu|}, \qquad Y_{|\nu|}(x) \to -\frac{\Gamma(|\nu|)}{\pi} \left(\frac{x}{2}\right)^{-|\nu|}$$

$$H^{(1)}_{|\nu|}(x) = J_{|\nu|}(x) + iY_{|\nu|}(x)\,, \qquad x \propto k(-\tau)^{(2+np)/2}\,.$$

For $np > -2$, x decreases $\Leftrightarrow$ modes exit the effective horizon.

$\omega_k(\tau) \propto \sqrt{-\tau}\, Y_{|\nu|}$ is the late time dominant solution.

16. Power spectrum of scalar field perturbations

Using $\omega_k(\tau) \propto \sqrt{-\tau}\, Y_{|\nu|}$, $\mathcal{P}_u = \frac{k^3}{2\pi^2}|\omega_k|^2$ and $\mathcal{P}_\phi = D\mathcal{P}_u/a^2$

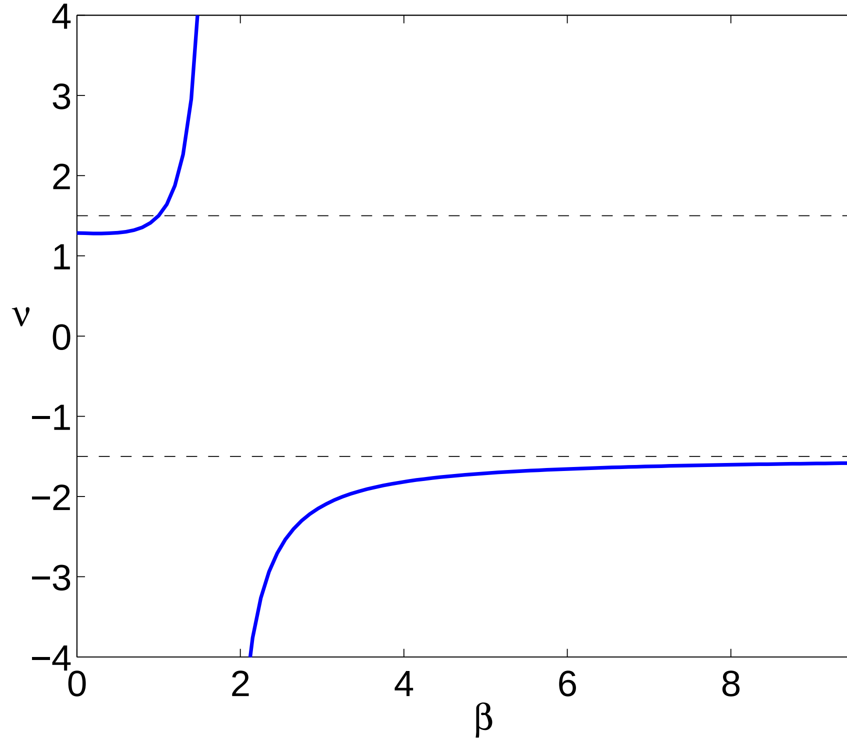
$$\mathcal{P}_\phi \propto \frac{H^2}{\sqrt{D}}\left(\frac{\sqrt{D}\,k}{aH}\right)^{3-2|\nu|} \propto k^{3-2|\nu|}\,(-\tau)^{1+p(n-2)-|\nu|(np+2)}$$

Scaling solution:

$$\nu = -\frac{\sqrt{9-12p+8np-12p^2-4p^2n+2n^2p^2}}{2+np}$$

$$p = -\frac{2}{\beta(n-r)+2(2+r)}$$

Scale invariance for large β (small p).



17. Fast-roll parameters and scale invariance

Near scale invariance $\Rightarrow \qquad \Delta n_u = 3 - 2|\nu| \approx 0 \qquad \Rightarrow$

$$p = \frac{\alpha}{\bar{\epsilon} - 2\alpha(2+r)} = -\frac{2}{\beta(n-r) + 2(2+r)} \approx 0$$

Steep and *negative* potentials and *fast-roll* evolution

Expand Δn_u in terms of fast-roll parameters

$$\epsilon \equiv 1/2\bar{\epsilon} = \frac{S}{D}\left(\frac{V}{V_{,\phi}}\right)^2$$

$$\eta \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2} - \frac{1}{2}\frac{V}{V_{,\phi}}\left(\frac{D_{,\phi}}{D} - \frac{S_{,\phi}}{S}\right)$$

and admitting that $\bar{\epsilon}$ is time dependent, the spectral index gives

$$\Delta n_u \approx 4\epsilon\left[1 - \frac{n}{12}\left(1 + \frac{n}{6} - r\right) - \frac{r}{2}\right] - 4\eta$$

Scale invariance is obtained for $\epsilon \approx 0$ and $\eta \approx 0$.

18. Quadratic corrections

Using holonomies as basic variables leads to a quadratic energy density contribution in the Friedmann equation

$$H^2 = \frac{1}{3}\rho\left(1 - \frac{\rho}{2\sigma}\right)$$

with $\rho < 2\sigma$. In this work we consider

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

The variation of the Hubble rate is

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\left(1 - \frac{\rho}{\sigma}\right)$$

Super-inflation for $\sigma < \rho < 2\sigma$.

19. Scaling solution (quadratic corrections)

"Scaling solution" $\Leftrightarrow$ $\dot{\phi}^2/(2\sigma - V) \approx$ cnst.

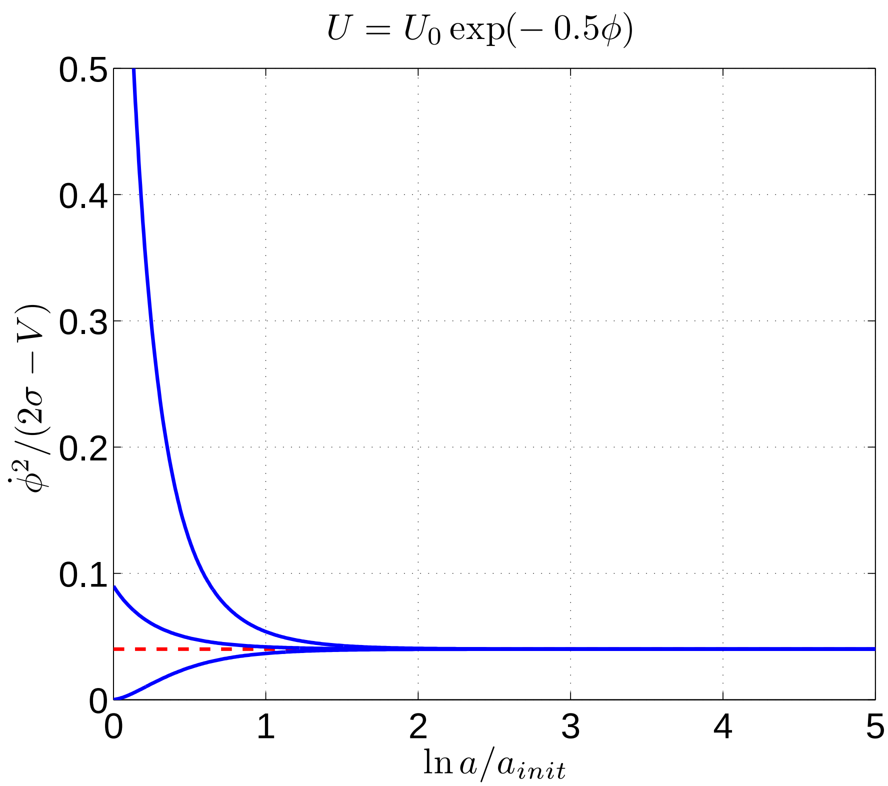
$$a = (-\tau)^p$$

$$p = -\frac{1}{\bar{\epsilon}+1}$$

$$\bar{\epsilon} = \frac{1}{2}\left(\frac{U_{,\phi}}{U}\right)^2$$

$$V = 2\sigma - U(\phi)$$

$$U = U_0\, e^{-\lambda\phi}$$



where $\lambda^2 = 2\bar{\epsilon}$.

Scaling solution is *stable* attractor for all λ or $\bar{\epsilon}$

20. Power spectrum of the perturbed field

Power spectrum is given by: $\mathcal{P}_u \propto k^3 \langle |\omega_k|^2 \rangle \propto k^{3-2|\nu|}(-\tau)^{1-2|\nu|}$

where $\nu = -\sqrt{1 - 4m_{\rm eff}^2\tau^2}/2$

For scaling solution $m_{\rm eff}^2\tau^2 = -2 + 3p(1+p)$

Near scale invariance $\Rightarrow$ $p = -\frac{1}{\bar{\epsilon}+1} = -\frac{2}{2\lambda^2+2} \approx 0$

Steep and *positive* potentials and *fast-roll* evolution

Expand Δn_u in terms of fast-roll parameters

$$\epsilon \equiv 1/2\bar{\epsilon} = \left(\frac{U}{U_{,\phi}}\right)^2 , \qquad \eta \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2}$$

and admitting that $\bar{\epsilon}$ is time dependent, the spectral index gives

$$\Delta n_u \approx -4(\epsilon - \eta)$$

Scale invariance is obtained for $\epsilon \approx 0$ and $\eta \approx 0$.

21. Number of e-folds and the horizon problem

Requirement that the scale entering the horizon today exited N e-folds before the end of inflation:

$$\ln\left(\frac{a_{\rm end}H_{\rm end}}{a_N H_N}\right) = 68 - \frac{1}{2}\ln\left(\frac{M_{\rm Pl}}{H_{\rm end}}\right) - \frac{1}{3}\ln\left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right)^{1/4}$$

1. In standard inflation: $\ln\left(\frac{a_{\rm end}H_{\rm end}}{a_N H_N}\right) \approx \ln\left(\frac{a_{\rm end}}{a_N}\right) \equiv N \approx 60$

2. In LQC with $a = (-\tau)^p$ and $p \ll 1$

$$\ln\left(\frac{a_{\rm end}H_{\rm end}}{a_N H_N}\right) = \ln\frac{\tau_N}{\tau_{\rm end}} = \ln\left(\frac{a_N}{a_{\rm end}}\right)^{1/p} = -\frac{1}{p}N$$

$$N \approx -60\,p$$

Number of e-folds of super-inflation required to solve the horizon problem can be of only a few.

22. Summary and questions

1. Inverse volume corrections: Scale invariance for steep negative potentials, $V = V_0\phi^{\beta}$;

2. Quadratic corrections: Scale invariance for steep positive potentials, $V = 2\sigma - U_0 \exp(-\lambda\phi)$;

3. Scaling solution is stable in both cases;

4. Only a few e-folds necessary to solve the horizon problem

5. What is the power spectrum of the curvature perturbation? Work in progress.