Quasi-local black hole horizons in Numerical Relativity: a quasi-equilibrium case

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IAP (Paris) May 22, 2006

Plan of the talk

1. Motivations for quasi-local black hole horizons : Isolated Horizons and boundary conditions in Numerical Relativity
2. Geometry of Isolated Horizons
   - Numerical relativitist's approach
   - Geometry of a null hypersurface : geometrical boundary conditions
   - Glimpse on the symplectic geometry : physical parameters
3. Analytical aspects : Conformal Thin Sandwich decompositions
4. Numerical implementations
5. Future work and Conclusions

Motivations

Motivations for quasi-local Black Hole horizons

General motivations

Alternative to the global *event horizon* notion : conceptual and technical

- Numerical Relativity : absence of global information during the evolution
- Black Hole Thermodynamics : laws extension beyond stationarity
- Quantum Gravity : microscopic understanding of Black Hole Entropy
- Mathematical Relativity : dynamics of trapped surfaces (Penrose conjecture), mass of solitonic solutions in Einstein-Yang-Mills theory

Motivations from 3+1 Numerical Relativity

Control of the BH characterization during the evolution

a) Calculation of physical parameters (M, J) (*a posteriori* analysis)

b) Key element in the resolution of the relevant PDE (*a priori* analysis)

Problem here : Boundary conditions on an *excised* sphere representing the BH horizon

Quasi-local BHs in a fully-constrained evolution scheme

Bonazzola et al. PRD **70** 104007 (2004)

$$\left.\begin{array}{r}\text{GR Constraints solved at each time step}\\ \text{Prescription on } \dot{K}\end{array}\right\} \Longrightarrow \boxed{\text{Five coupled elliptic eqs.}}$$

Rest of the fields :

- Evolution : hyperbolic equations for the propagating modes
  (Dirac) gauge $\rightarrow$ two physical degrees of freedom
- Initial Data : choice of free initial data

In this talk...

We focus on the construction of initial data in quasi-equilibrium

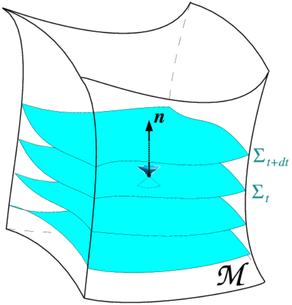
$\Longrightarrow$ Isolated Horizons

Motivations/objectives :

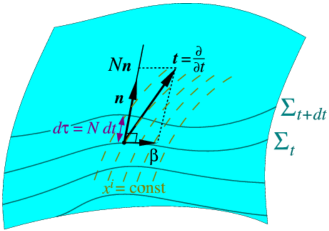
- Construction of Binary Black Hole initial data : gravitational waves physics
- Warming-up exercise before full evolution of BHs

Geometrical aspects

Geometric inner boundary conditions : 3+1 notation



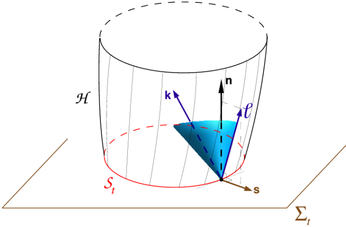
$\{\Sigma_t\}$	3+1 slicing of spacetime
n^μ	timelike unit normal to Σ_t
$t^\mu = N n^\mu + \beta^\mu$	evolution vector
N	lapse function
β^μ	shift vector
$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$	spatial 3-metric
$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$	extrinsic curvature

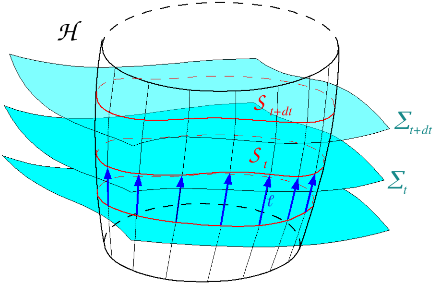


Numerical relativitist's approach to the inner geometrical boundary conditions

Basic notion : apparent horizon $\mathcal{S}_t$

s^μ	unit normal vector to $\mathcal{S}_t$, in Σ_t
ℓ^μ	outgoing null vector
k^μ	ingoing null vector ($k^\mu \ell_\mu = -1$)
$q_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu$	induced metric on $\mathcal{S}_t$
$\theta_{(\ell)} \equiv q^{\mu\nu}\nabla_\mu \ell_\nu = 0$	Vanishing (outgoing) expansion (apparent horizon condition)





World-tube $\mathcal{H}$ of apparent horizons $\mathcal{S}_t$

$\mathcal{S}_t$ constant area $\Rightarrow \mathcal{H}$ null hypersurface

$\mathcal{H}$ generated by ℓ^μ : outgoing null vector

Given the induced slicing $\{\mathcal{S}_t\} \Longrightarrow$

Natural evolution vector on $\mathcal{H}$:

$$\boxed{\ell = N \cdot (n^\mu + s^\mu)}$$

(ℓ Lie draggs the surfaces $\mathcal{S}_t$)

Geometric inner boundary conditions

Quasi-equilibrium : *time independence* of certain 3+1 fields

1) Metric $q_{\mu\nu}$: $\mathcal{L}_\ell q_{\mu\nu} = 0 \Leftrightarrow q^\rho{}_\mu q^\sigma{}_\nu \nabla_\rho \ell_\sigma = 0$

trace (expansion) : $\boxed{\theta_{(\ell)} = 0}$

trace-free (shear) : $q^\rho{}_\mu q^\sigma{}_\nu \nabla_\rho \ell_\sigma - \frac{1}{2}\theta_{(\ell)} q_{\mu\nu} \equiv \boxed{\left(\sigma_{(\ell)}\right)_{\mu\nu} = 0}$

Actual restriction to the geometry

2) *Normal-tangent* components of the extrinsic curvature : $K_{\rho\sigma} q^\rho{}_\mu s^\sigma$

$$\mathcal{L}_\ell \underbrace{(K_{\rho\sigma} q^\rho{}_\mu s^\sigma)}_{\sim\Omega_\mu} = 0 \Rightarrow \exists \text{ function } \kappa \text{ on } \mathcal{S}_t \text{ such that :}$$

$$\boxed{{}^2D\kappa = 0}$$

where ${}^2D$ is the connection associated with $q_{\mu\nu}$.

3) Lapse N : $\boxed{\mathcal{L}_\ell N = 0}$

Gauge condition : choice of a *coordinate system adapted to the horizon* $\mathcal{H}$

constant coordinate radius $r = \mathrm{const} \iff t^\mu$ tangent to $\mathcal{H}$

Writing : $\beta^\mu = b s^\mu - V^\mu$, with $V^\mu s_\mu = 0$

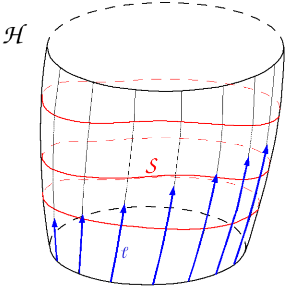
$$t^\mu = \ell^\mu + (b - N) s^\mu - V^\mu \quad \Longrightarrow \quad \boxed{b - N = 0}$$

Others...

- Analytical well-posedness... (conditions on $\Psi^6 \cdot K_{\mu\nu} s^\mu s^\nu$)
- Numerical control of the slicing, taking into account the horizon geometry... (conditions on the lapse N)

Need of a systematic treatment !

Geometrical approach : Isolated horizons



Ashtekar and Krishnan, Liv.Rev.Rel **7**, 10 (2004)

Non-expanding horizon

- Null-hypersurface $\mathcal{H} \approx S^2 \times \mathbb{R}$ sliced by marginally (outer) trapped surfaces $\mathcal{S}$: $\theta_{(\ell)} = 0$.
  Raychaudhuri equation $\Rightarrow \sigma_{(\ell)} = 0$
- Einstein equations satisfied on $\mathcal{H}$
- $-T^{\mu}{}_{\nu}\ell^{\nu}$ future directed

Well defined connection $\hat{\nabla}$, induced by the spacetime ∇ :
Geometry of the null hypersurface $\mathcal{H}$ charaterized by $(q_{\mu\nu}, \hat{\nabla})$

- Some components of $\hat{\nabla}$ define an intrinsic 1-form ω on $\mathcal{H}$:
$$\hat{\nabla}_{\mu}\ell^{\nu} = \omega_{\mu}\ell^{\nu}$$
- Notion of surface gravity : $\hat{\nabla}_{\ell}\ell^{\mu} = \kappa_{(\ell)}\ell^{\mu} \quad \Leftrightarrow \quad \boxed{\kappa_{(\ell)} = \ell^{\mu}\omega_{\mu}}$

Isolated horizons : hierarchical structure

Physical idea : dynamical spacetime with a black hole in equilibrium
Isolated Horizon hierarchy : increasing level of equilibrium

- Non-Expanding Horizon (NEH) : $\boxed{\mathcal{L}_\ell \, q_{\mu\nu} = 0}$

  minimal constraint on the geometry

- Weakly Isolated Horizon (WIH) : $\boxed{\mathcal{L}_\ell \, \omega_\mu = 0}$

  Dependent on ℓ due to the rescaling behaviour :

$$\ell \to \ell' = \alpha \ell \quad \Longrightarrow \omega \to \quad \omega + \hat{\nabla}\alpha$$

  Restriction of ℓ to a WIH-equivalence class : $\ell \sim \ell'$ iff $\ell' = \mathrm{const} \cdot \ell$

  $\boxed{\text{WIH} = \text{NEH} + \text{WIH-equivalence class of null normals}}$

  Not a restriction on the null geometry ! (see later...)

- (Strongly) Isolated Horizon : $\boxed{[\mathcal{L}_\ell, \hat{\nabla}] = 0}$

  Strongest equilibrium condition on the geometry

Geometrical consequences

NEH

$$\left.\begin{array}{l}\theta_{(\ell)} = 0\\ \sigma_{(\ell)} = 0\end{array}\right\} \Longrightarrow \mathcal{L}_\ell q_{\mu\nu} = 0$$

In addition, for the components of the Weyl tensor :

$$d\omega = \mathrm{Im}\Psi_2\, {}^2\epsilon \quad ; \quad \Psi_0 = 0 = \Psi_1$$

WIH

$$\mathcal{L}_\ell \omega = 0 \Leftrightarrow \hat{\nabla}\kappa_{(\ell)} = 0 \text{ (\textit{zeroth law of BH mechanics})}$$

If $\kappa_{(\ell)} \neq \mathrm{const}$, then $\ell' = \alpha\ell$, with $\mathrm{const} = \nabla_\ell \alpha + \alpha\kappa_\ell$, has const $\kappa_{(\ell')}$.

Therefore, a WIH is not a restriction on a NEH.

It is rather a condition on the null normal $\ell \Leftrightarrow$ the 3+1 slicing

$\boxed{\text{WIH-compatible slicings}}$

IH

Mass and angular momentum *multipole moments* characterizing the horizon $\mathcal{H}$

3+1 expressions

We introduce a 3+1 slicing (*arbitrary but fixed*)

Null normals : $\boldsymbol{\ell} = N(\boldsymbol{n} + \boldsymbol{s})$ and $\boldsymbol{k} = \frac{1}{2N}(\boldsymbol{n} - \boldsymbol{s})$

2+1 decomposition

$$\omega_\mu = \Omega_\mu - \kappa_{(\ell)} k_\mu$$

$$\Xi_{\mu\nu} \equiv q^\rho{}_\mu q^\sigma{}_\nu \nabla_\mu k_\nu \qquad \left(= (\sigma_{(k)})_{\mu\nu} + \frac{1}{2}\theta_{(k)} q_{\mu\nu}\right)$$

$$\boxed{(q_{\mu\nu}, \hat{\nabla}) \Longleftrightarrow (q_{\mu\nu}, \kappa, \Omega_\mu, \Xi_{\mu\nu})}$$

3+1 forms :

$$\begin{aligned} \Omega_\alpha &= {}^2D_\alpha \ln N - K_{\mu\nu} s^\mu q^\nu{}_\alpha \\ \kappa &= \ell^\mu \nabla_\mu \ln N + s^\mu D_\mu N - N K_{\mu\nu} s^\mu s^\nu \\ \Xi_{\alpha\beta} &= -\frac{1}{2N}(D_\mu s_\nu + K_{\mu\nu}) q^\mu{}_\alpha q^\nu{}_\beta \end{aligned}$$

Quasi-equilibrium conditions (only involving first time derivatives...)

- $\mathcal{L}_\ell q_{\mu\nu} = 0 \Rightarrow \boxed{\theta_{(\ell)} = \sigma_{(\ell)} = 0}$
- $\mathcal{L}_\ell \Omega_\mu$: Navier-Stokes-like equation (*membrane paradigm*)

$$\begin{aligned}\underbrace{\partial_t \Omega_a + V^b\, {}^2D_b \Omega_a + \Omega_b\, {}^2D_a V^b}_{\mathcal{L}_\ell \Omega_a} + \theta\Omega_a \quad &= \quad 8\pi q^\mu{}_a T_{\mu\nu}\, \ell^\nu + {}^2D_a \kappa \\ &- \quad {}^2D_b \sigma^b{}_a + \frac{1}{2}{}^2D_a \theta\end{aligned}$$

(Gourgoulhon PRD **72** (2005) 104007)

$$\theta = 0 = \sigma_{ab} \Rightarrow \boxed{\mathcal{L}_\ell \Omega_a = {}^2D_a \kappa} \qquad \textit{pressure}\ \text{gradient}$$

Consequence : Evolution equation for the lapse N on $\mathcal{H}$ (with $\kappa = \kappa_o = \mathrm{const}$) :

$$\boxed{\mathcal{L}_\ell \ln N = \kappa_o - s^\mu D_\mu N + N K_{\mu\nu} s^\mu s^\nu}$$

If we add $\mathcal{L}_\ell N = 0$,

$$\boxed{\kappa_o = s^\mu D_\mu N - N K_{\mu\nu} s^\mu s^\nu}$$

Quasi-equilibrium conditions II

- $\mathcal{L}_\ell\, \Xi_{\mu\nu}$: vanishing $\Rightarrow$ all geometric information encoded in $(q_{\mu\nu}, \Omega_\mu)$

$$\boldsymbol{\mathcal{L}_\ell\, \Xi} = \frac{1}{2}\,\mathrm{Kil}(\boldsymbol{{}^2D}, \boldsymbol{\Omega}) + \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} - \frac{1}{2}{}^2\boldsymbol{R} + 4\pi\left(\vec{\boldsymbol{q}}^*\boldsymbol{T} - \frac{T}{2}\boldsymbol{q}\right) - \kappa\boldsymbol{\Xi}$$

Then, $\boldsymbol{\mathcal{L}_\ell\, \Xi} = 0$ implies $(\kappa \neq 0)$:

$$\boxed{\kappa\boldsymbol{\Xi} = \frac{1}{2}\,\mathrm{Kil}(\boldsymbol{{}^2D}, \boldsymbol{\Omega}) + \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} - \frac{1}{2}{}^2\boldsymbol{R} + 4\pi\left(\vec{\boldsymbol{q}}^*\boldsymbol{T} - \frac{T}{2}\boldsymbol{q}\right)}$$

We will not discuss this condition in this talk. However, it is very relevant when solving the evolution equations for the physical radiative degrees of freedom

Intrinsic determination of the foliation

On a non-extremal WIH ($\kappa \neq 0$),

$$\boxed{\text{fixing } (\mathcal{S}_t) \Leftrightarrow \text{fixing ingoing null vector } \boldsymbol{k} \Leftrightarrow \text{fixing the 1-form } \boldsymbol{\Omega} \text{ on } \mathcal{S}_t}$$

Hodge decomposition on S^2 of $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\rm div-free} + \boldsymbol{\Omega}^{\rm exact}$$

with ${}^{\boldsymbol{2}}\boldsymbol{D} \cdot \boldsymbol{\Omega}^{\rm div-free} = 0$ and $\boldsymbol{\Omega}^{\rm exact} = {}^{\boldsymbol{2}}\boldsymbol{D} f$ for some function f on S^2.

- Divergence-free part :

$$\boldsymbol{d}\boldsymbol{\Omega}^{\rm div-free} = 2 \, \mathrm{Im} \, \Psi_2 {}^2\boldsymbol{\epsilon}$$

- Exact part :

$${}^2\Delta f = {}^{\boldsymbol{2}}\boldsymbol{D} \cdot \boldsymbol{\Omega}^{\rm exact} = {}^{\boldsymbol{2}}\boldsymbol{D} \cdot \boldsymbol{\Omega} \equiv g$$

Therefore :

$$\boxed{{}^2\Delta \ln N = {}^2D^\rho (q^\mu{}_\rho K_{\mu\nu} s^\nu) + g}$$

Physical Parameters

Physical parameter : conserved quantity under a *symmetry* transformation (*canonical transformation* on the solution (phase) space of Einstein equation)

Underlying symmetry notion :

WIH-symmetries

A vector field $\boldsymbol{W}$ on a WIH $(\mathcal{H}, [\ell])$ is a WIH-symmetry iff :

$$\mathcal{L}_{\boldsymbol{W}}\boldsymbol{\ell} = \mathrm{const}\cdot\boldsymbol{\ell}, \quad \mathcal{L}_{\boldsymbol{W}}\boldsymbol{q} = 0 \quad \text{and} \quad \mathcal{L}_{\boldsymbol{W}}\boldsymbol{\omega} = 0$$

General form of $\boldsymbol{W}$:

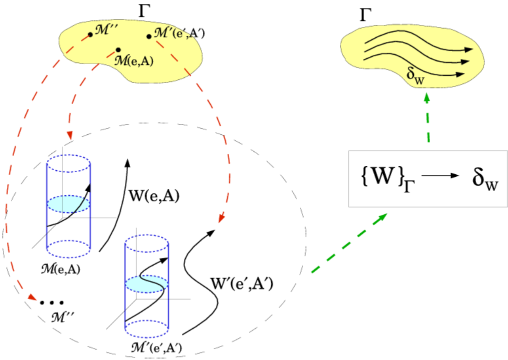
$$\boldsymbol{W} = c_{\boldsymbol{W}}\boldsymbol{\ell} + b_{\boldsymbol{W}}\boldsymbol{S}$$

where $c_{\boldsymbol{W}}$ and $b_{\boldsymbol{W}}$ are constants and $\boldsymbol{S}$ is a symmetry of $\mathcal{S}_t$.

Physical Parameters : symplectic (hamiltonian) analysis

Procedure

1) Construction of the phase space Γ (each point a spacetime $\mathcal{M}$)
2) Extension of $\boldsymbol{W}$ on $\mathcal{H}$ to infinitesimal diffeomorphism on each $\mathcal{M}$ → family $\{\boldsymbol{W}\}_\Gamma$
3) $\{\boldsymbol{W}\}_\Gamma \to$ canonical transformation $\delta_{\boldsymbol{W}}$ on Γ ($\delta_{\boldsymbol{W}}$ preserves the symplectic form)
4) Physical parameter : conserved quantity under $\delta_{\boldsymbol{W}}$



Physical Parameters I : angular momentum and mass

Angular momentum

ϕ^μ axial symmetry on $\mathcal{S}_t$ $\rightarrow$ $\delta_{\boldsymbol{\phi}}$ canonical transformation

$$J_{\mathcal{H}} = -\frac{1}{8\pi G}\int_{\mathcal{S}_t} \omega_\mu \phi^\mu \; {}^2\boldsymbol{\epsilon} = -\frac{1}{4\pi G}\int_{\mathcal{S}_t} f \mathrm{Im}\Psi_2 {}^2\boldsymbol{\epsilon}$$

with $\boldsymbol{\phi} = {}^2\vec{\boldsymbol{D}}f \cdot {}^2\boldsymbol{\epsilon}$ (since $\boldsymbol{\phi}$ is divergence-free)

$$J_{\mathcal{H}} = -\frac{1}{8\pi G}\int_{\mathcal{S}_t} \Omega_\mu \phi^\mu \; {}^2\boldsymbol{\epsilon} = \frac{1}{8\pi G}\int_{\mathcal{S}_t} \phi^\mu s^\nu K_{\mu\nu} \; {}^2\boldsymbol{\epsilon}$$

Physical Parameters II : angular momentum and mass

Mass : *1st law of black hole thermodynamics*

Evolution vector $\boldsymbol{t} = \boldsymbol{\ell} + \Omega_{(\boldsymbol{t})}\boldsymbol{\phi}$.

1. Transformation $\delta_{\boldsymbol{t}}$ canonical iff $\exists\ E^t_{\mathcal{H}}$:

$$\delta E^t_{\mathcal{H}} = \frac{\kappa_{(\boldsymbol{t})}(a_{\mathcal{H}}, J_{\mathcal{H}})}{8\pi G}\,\delta a_{\mathcal{H}} + \Omega_{(\boldsymbol{t})}(a_{\mathcal{H}}, J_{\mathcal{H}})\,\delta J_{\mathcal{H}}$$

with $a_{\mathcal{H}} = \int_{\mathcal{S}_t} {}^2\boldsymbol{\epsilon} = 4\pi R^2_{\mathcal{H}}$ the area of $\mathcal{S}_t$.

Additional motivation for $\kappa = \mathrm{const}$ condition !

2. Normalization of the energy function : stationary Kerr family $(a_{\mathcal{H}}, J_{\mathcal{H}})$

$$\begin{array}{rcl} M_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) & := & M_{\mathrm{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = \frac{\sqrt{R^4_{\mathcal{H}} + 4G^2J^2_{\mathcal{H}}}}{2GR_{\mathcal{H}}}\ , \\ \kappa_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) & := & \kappa_{\mathrm{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = \frac{R^4_{\mathcal{H}} - 4G^2J^2_{\mathcal{H}}}{2R^3_{\mathcal{H}}\sqrt{R^4_{\mathcal{H}} + 4GJ^2_{\mathcal{H}}}}\ , \\ \Omega_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) & := & \Omega_{\mathrm{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = \frac{2GJ_{\mathcal{H}}}{R_{\mathcal{H}}\sqrt{R^4_{\mathcal{H}} + 4GJ^2_{\mathcal{H}}}} \end{array}$$

Analytical aspects

Analytical aspects : conformal decompositions

Conformal Thin Sandwich approach to Initial Data

Conformal decomposition of (γ_{ij}, K^{ij}) on $\Sigma_t \sim \mathbb{R}^3 \setminus S^2$:

- 3-metric

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$$

  with $\tilde{\gamma}$ unimodular : $\det(\tilde{\gamma}_{ij}) = \det(f_{ij})$ (f_{ij} background flat metric)

- Extrinsic curvature

$$K_{ij} = \Psi^\zeta \tilde{A}_{ij} + \frac{1}{3} K \gamma_{ij}$$

  where

$$\tilde{A}^{ij} = \frac{\Psi^{4-\zeta}}{2N} \left(\tilde{D}^i \beta^j + \tilde{D}^j \beta^i - \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij} + \dot{\tilde{\gamma}}^{ij} \right)$$

Analytical aspects : coupled PDE system

Hamiltonian constraint :

$$\tilde{D}_k \tilde{D}^k \Psi - \frac{{}^3\tilde{R}}{8}\,\Psi + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\,\Psi^{2\zeta-3} + \left(2\pi E - \frac{K^2}{12}\right)\Psi^5 = 0$$

Momentum constraint :

$$\begin{aligned}\tilde{D}_k \tilde{D}^k \beta^i + \frac{1}{3}\tilde{D}^i \tilde{D}_k \beta^k + {}^3\tilde{R}^i{}_k \beta^k \quad &= \quad 16\pi\Psi^4 N J^i + \frac{4}{3} N \tilde{D}^i K - \tilde{D}_k \dot{\tilde{\gamma}}^{ij} \\ &\quad + 2N\Psi^{\zeta-4}\tilde{A}^{ik}\tilde{D}_k \ln(N\Psi^{-6})\end{aligned}$$

Prescription for $\dot{K}$ (part of the gauge freedom)

$$\begin{aligned}\tilde{D}_k \tilde{D}^k N + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N \quad &= \quad \Psi^4 \left\{ N\left[4\pi(E+S) + \frac{K^2}{3}\right] \right. \\ &\qquad \left. - \dot{K} + \beta^k \tilde{D}_k K \right\} + N\Psi^{2\zeta-4}\tilde{A}_{kl}\tilde{A}^{kl}\end{aligned}$$

Analytical aspects : coupled PDE system II

Remarks

- Coupled non-linear elliptic system on (Ψ, β^i, N)
  Possibility of redefining/rescaling the fields in order to improve analytical behaviour (*maximum principle*...) : $N = \tilde{N}\Psi^a$
- Free initial data : $(\tilde{\gamma}_{ij}, \dot{\tilde{\gamma}}_{ij}, K, \dot{K})$ *and* the boundary conditions on the inner sphere S^2 for $(\Psi, \beta^i, N)|_{S^2}$.
- In the fully-constrained scheme proposed in Bonazzola et al. (2004) :

  a) Same system of coupled elliptic equations
  b) Additional evolution equations for $\tilde{\gamma}_{ij}$
  Proposed choice of gauges : maximal slicing ($K = 0$) and generalized *Dirac gauge* ($\mathcal{D}_k\tilde{\gamma}^{ki} = 0$)

Analytical aspects : re-scaled coupled PDE

Rescaling : $N = \tilde{N}\psi^a$

- $$\tilde{\Delta}\Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{32}\Psi^{5-2a}\tilde{N}^{-2}(\tilde{L}\beta)_{ij}(\tilde{L}\beta)^{ij} - \frac{1}{12}K^2\Psi^5 = 0,$$

- $$\tilde{\Delta}\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_k\beta^k + \tilde{R}^i_k\beta^k - \tilde{N}^{-1}(\tilde{L}\beta)^{ik}\tilde{D}_k\tilde{N}$$
$$-(a-6)\Psi^{-1}(\tilde{L}\beta)^{ik}\tilde{D}_k\Psi = \frac{4}{3}\Psi^a\tilde{N}\tilde{D}^i K \quad ,$$

- $$\tilde{\Delta}\tilde{N} + 2(a+1)\tilde{D}^k \ln\Psi \tilde{D}_k \ln\tilde{N}$$
$$+\tilde{N}\left[\frac{a}{8}\tilde{R} + \frac{a-4}{12}\Psi^4K^2 + a(a+1)\tilde{D}^k\ln\Psi\tilde{D}_k\ln\Psi\right]$$
$$-\frac{a+8}{32}\Psi^{4-2a}\tilde{N}^{-1}(\tilde{L}\beta)_{ij}(\tilde{L}\beta)^{ij} = \Psi^{4-a}\beta^k\tilde{D}_kK \quad .$$

No obvious (...possible ?) choice of a for applying a maximum principle...

Completing the elliptic system : inner boundary conditions I

Constrained functions

- Conformal factor : Ψ
- Shift :

$$\beta^i = \tilde{b}\tilde{s}^i - V^i \longrightarrow \begin{cases} \tilde{b} & \text{radial part of the shift} \\ V^i & \text{part of the shift tangent to } \mathcal{S} \end{cases}$$

- Lapse : N

1. *Apparent Horizon* boundary condition : $\theta_{(l)} = 0$.

$$\boxed{4\tilde{s}^i\tilde{D}_i\Psi + \tilde{D}_i\tilde{s}^i\Psi + \Psi^{-1}K_{ij}\tilde{s}^i\tilde{s}^j - \Psi^3 K = 0}$$

Note : sign of $\tilde{s}^i\tilde{D}_i\Psi$ depends on the sign and size of $K_{ij}\tilde{s}^i\tilde{s}^j \sim \tilde{s}^i\tilde{D}_i\tilde{b} + \ldots$
(constraints on $\tilde{s}^i\tilde{D}_i\tilde{b}$ if maximum principle argument...)

Completing the elliptic system : inner boundary conditions II

2. Quasi-equilibrium condition : $\sigma_{ab} = 0$.

$$0 = \sigma_{ab} = \underbrace{\left(\partial_t \tilde{q}_{ab} - \frac{1}{2}\left(\partial_t \ln \tilde{q}\right)\tilde{q}_{ab}\right)}_{\text{initial free data}} + \underbrace{\left({}^2\tilde{D}_a \tilde{V}_b + {}^2\tilde{D}_b \tilde{V}_a - \left({}^2\tilde{D}_c V^c\right)\tilde{q}_{ab}\right)}_{\text{intrinsic geometry of } \mathcal{S}_t}$$

$$+ \underbrace{\left(\Psi^{-2}N - \tilde{b}\right)\left(\tilde{H}_{ab} - \frac{1}{2}\tilde{q}_{ab}\tilde{H}\right)}_{\text{“extrinsic” geometry of } \mathcal{S}_t}$$

$$\left(\begin{array}{l} \tilde{q}_{ab} \text{ induced } \textit{conformal} \text{ metric on } \mathcal{S} : \quad \tilde{q}_{ij} = \tilde{\gamma}_{ij} - \tilde{s}_i \tilde{s}_j \\ \\ \tilde{H}_{ab} \text{ extrinsic curvature of } \mathcal{S} \text{ in } \Sigma_0 \end{array}\right)$$

$$\boxed{V^i \text{ conformal symmetry of } \mathcal{S}. \quad \text{Ex. : } V^i = \Omega(\partial_\varphi)^i}$$

Completing the elliptic system : inner boundary conditions III

3. Coordinate system adapted to the horizon : $b = N \Longrightarrow \boxed{\tilde{b} = N\Psi^{-2}}$

Loss of control on $\tilde{s}^i \tilde{D}_i \tilde{b}$...

4. Well-posedness of the elliptic system (CTT : no equation for N)

$$-\tilde{D}_i \tilde{s}^i < \underbrace{\Psi^6 \cdot K_{ij} s^i s^j}_{g} \leq 0$$

S. Dain, JLJ, Krishnan, Phys. Rev. D **71**, 064003 (2005)

In terms of 3+1 fields :

$$\boxed{2\tilde{s}^i \tilde{D}_i \tilde{b} - \tilde{b}\tilde{H} = 3N\Psi^{-6} g - {}^2\tilde{D}_a V^a - 2V^i \tilde{s}^j \tilde{D}_j \tilde{s}_i - NK}$$

Geometrical interpretation of the sign : *future* trapped surfaces $\theta_{(k)} \leq 0$

$$K_{ij} s^i s^j - K = \frac{1}{2N}\theta_{(\ell)} + N\theta_{(\boldsymbol{k})} \leq 0$$

Very important in the dynamical case !

Summary of boundary conditions

NEH b. c.	$\theta_{(\boldsymbol{\ell})} = 0$	$4\tilde{s}^i\tilde{D}_i\ln\Psi + \tilde{D}_i\tilde{s}^i + \Psi^{-2}K_{ij}\tilde{s}^i\tilde{s}^j - \Psi^2K = 0$
	$\boldsymbol{\sigma} = 0$	${}^2\tilde{\Delta}V^a + {}^2\tilde{R}^a{}_bV^b = {}^2\tilde{D}^b\tilde{C}_b{}^a$
Non-eq. b. c.	$r = \mathrm{const}$	$\tilde{b} = N\Psi^{-2}$
	$K_{ij}s^is^j = h_1$	$2\tilde{s}^k\tilde{D}_k\tilde{b} - \tilde{b}\tilde{H} = 3Nh_1 - {}^2\tilde{D}_kV^k - 2V^k\tilde{D}_{\tilde{s}}\tilde{s}_k - NK$
WIH b. c.	${}^{\mathcal{H}}\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}N = h_2$	$\kappa_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = s^iD_iN - NK_{ij}s^is^j + h_2$
	${}^{\mathcal{H}}\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\theta_{(\boldsymbol{k})} = h_3$	${}^2D^\mu{}^2D_\mu N - 2K_{\mu\nu}s^\nu\,{}^2D^\mu N + \left(-{}^2D^\rho(q^\mu{}_\rho K_{\mu\nu}s^\nu) + q^{\mu\rho}(K_{\mu\nu}s^\nu)(K_{\rho\sigma}s^\sigma) - \frac{1}{2}{}^2R + \frac{1}{2}q^{\mu\nu}R_{\mu\nu}\right)N + \frac{\kappa_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}})}{2}(D_\mu s^\mu - K_{\mu\nu}s^\mu s^\nu + K) = Nh_3$
	${}^2\boldsymbol{D}\cdot\boldsymbol{\Omega} = h_4$	${}^2\Delta\ln N = {}^2D^\rho(q^\mu{}_\rho K_{\mu\nu}s^\nu) + h_4$

Cook, Phys. Rev. D **65**, 084003 (2002) ; Cook, Pfeiffer, Phys. Rev. D **70**, 104016 (2004)

JLJ, Gourgoulhon, Mena Marugán, Phys. Rev. D **70**, 124036 (2004)

Ansorg, Phys. Rev. D **72**, 024018 (2005).

Numerical aspects

Numerical methods

Spectral methods

- Expansion of the functions on a truncated basis of orthogonal polynomials (Tchebychev polynomials...) : information encoded in the spectral coefficients
- Specially successful for elliptic equations

LORENE C++ library

- Specially adapted for spherical coordinates
- Multi-domain : kernel, shells, external domain.
- Compactified external domain (infinity...)
- Iterative scheme : non-linear and non-flat terms treated as sources at each iteration step (*passed to the right-hand-side*...)

Numerical implementation (I)

We keep fixed :

$$\begin{cases} 4\tilde{s}^i\tilde{D}_i\Psi + \tilde{D}_i\tilde{s}^i\Psi + \Psi^{-1}K_{ij}\tilde{s}^i\tilde{s}^j - \Psi^3K = 0 \\ \boldsymbol{V} = \Omega\cdot\partial_\varphi \end{cases}$$

And combine :

$$\begin{array}{ll} (b1) & \tilde{b} = N\Psi^{-2} \\ (b2) & 2\tilde{s}^i\tilde{D}_i\tilde{b} - \tilde{b}\tilde{H} = f - {}^2\tilde{D}_aV^a - 2V^i\tilde{s}^j\,\tilde{D}_j\tilde{s}_i - NK \end{array}$$

$$\begin{array}{ll} (NG1) & s^iD_iN - s^is^jK_{ij}\,N =|_{\mathcal{S}}\ \kappa_{_{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) \\ (NG2) & {}^2\Delta\ln N = {}^2D^\rho(q^\mu{}_\rho K_{\mu\nu}s^\nu) + g \end{array}$$

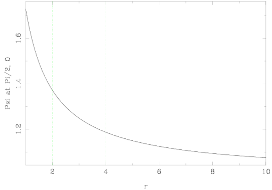
Effective boundary conditions

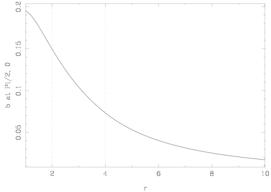
$$\begin{array}{llll} (NE0) & N = \text{const} = 0.2 & & \\ (NE1) & N\Psi = \frac{1}{2} & (NE2) & N\Psi = \frac{1}{\sqrt{2}}\Psi_{_{Kerr-Schild}} \\ (NE3) & \partial_r\,(N\Psi) = 0 & (NE4) & \partial_r\,(N\Psi) = \frac{N\Psi}{2r} \end{array}$$

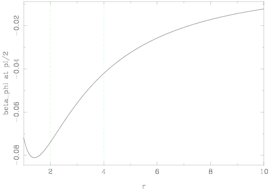
Numerical implementation (II) : (b2,NG1)

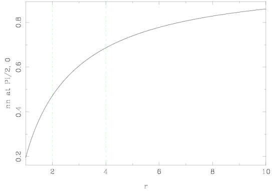
2 shells and an external zone compactified at infinity :

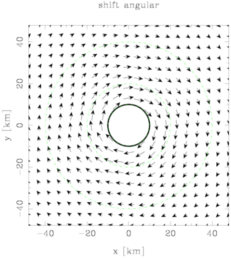
$nr \times n\theta \times n\varphi = 25 \times 17 \times 16$

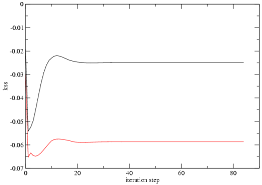








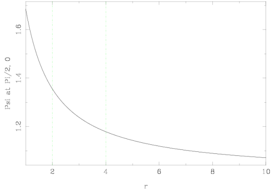


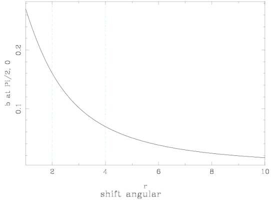


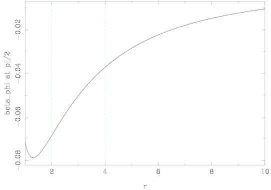
Numerical implementation (III) : (b1,NG2)

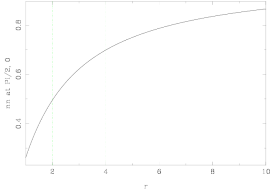
2 shells and an external zone compactified at infinity :

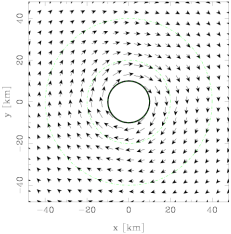
$nr \times n\theta \times n\varphi = 25 \times 17 \times 16$

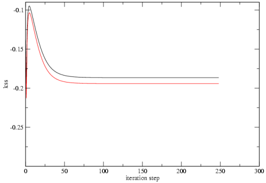








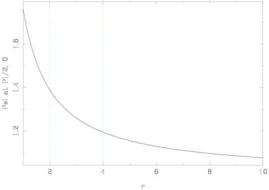


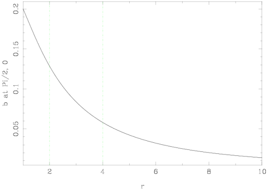


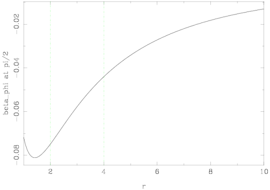
Numerical implementation (IV) : (b1,NE0)

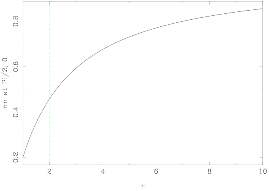
2 shells and an external zone compactified at infinity :

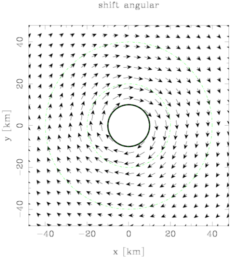
$nr \times n\theta \times n\varphi = 25 \times 17 \times 16$

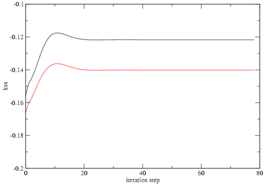








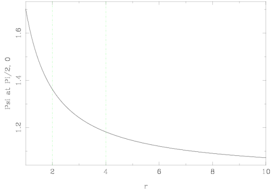


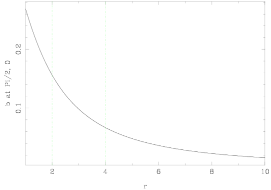


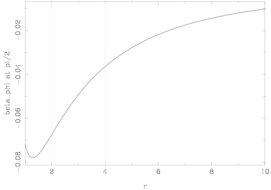
Numerical implementation (V) : (b2,NE1)

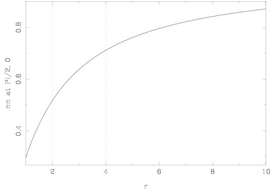
2 shells and an external zone compactified at infinity :

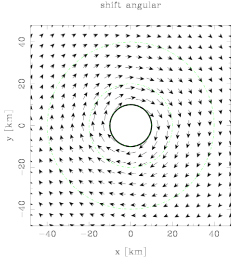
$nr \times n\theta \times n\varphi = 25 \times 17 \times 16$

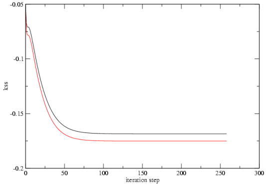










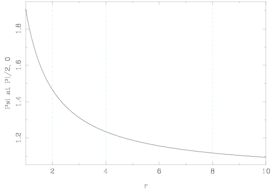


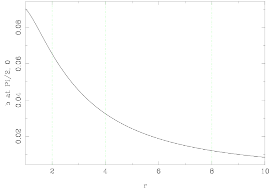
Numerical implementation (VI) : (b2,NG1)

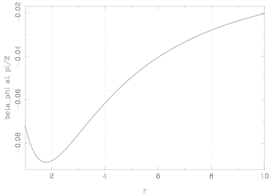
2 shells and an external zone compactified at infinity :

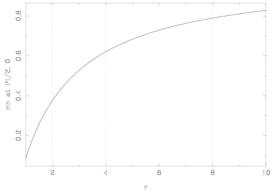
$nr \times n\theta \times n\varphi = 17 \times 9 \times 8$

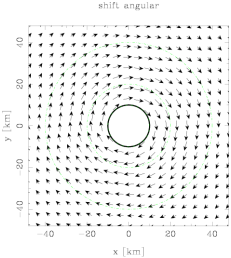
$K_{ij} s^i s^j = -0.05$

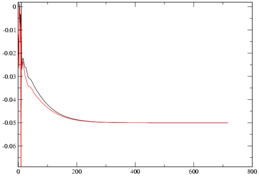








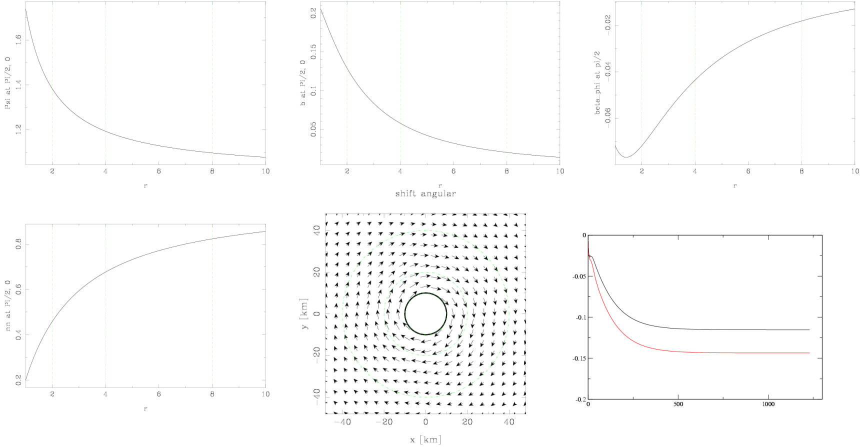




Numerical implementation (VII) : (b2,NE0)

2 shells and an external zone compactified at infinity :

$nr \times n\theta \times n\varphi = 17 \times 9 \times 8$ $K_{ij} s^i s^j = \frac{1}{N}\left(s^i D_i N - \kappa_{Kerr}(R_{\mathcal{H}}, J_{\mathcal{H}})\right)$



Technical important results

- Degeneracy of $\theta_{(\ell)} = 0 = \sigma_{(\ell)}, b = N, \kappa = \mathrm{const}$?
  1) Not denegerated when $\kappa = \kappa_o = \mathrm{const}$ is prescribed !
  2) No solution if $\kappa = \kappa_{\mathrm{Kerr}}(a, J)$
     (or degenerated in the non-rotating case)

  Preferred set of inner boundary conditions for the CTS equations...

- $K_{ij}s^i s^j \cdot \Psi^6$ good boundary condition in CTS ?
  In general, not a well-posed question... need to especify N.
  For *generic* N :
  a) $K_{ij}s^i s^j \cdot \Psi^6$ is in fact bounded by below (also in the CTS).
  b) The physical $K_{ij}s^i s^j$ (and therefore $\theta_{(k)}$!), is not.

  Conclusion :The good parameter to be imposed as boundary condition is in fact the physical $\theta_{(k)}$ ($K_{ij}s^i s^j \cdot \Psi^6$ leads to non-unique solutions)

Future (current !) perspectives

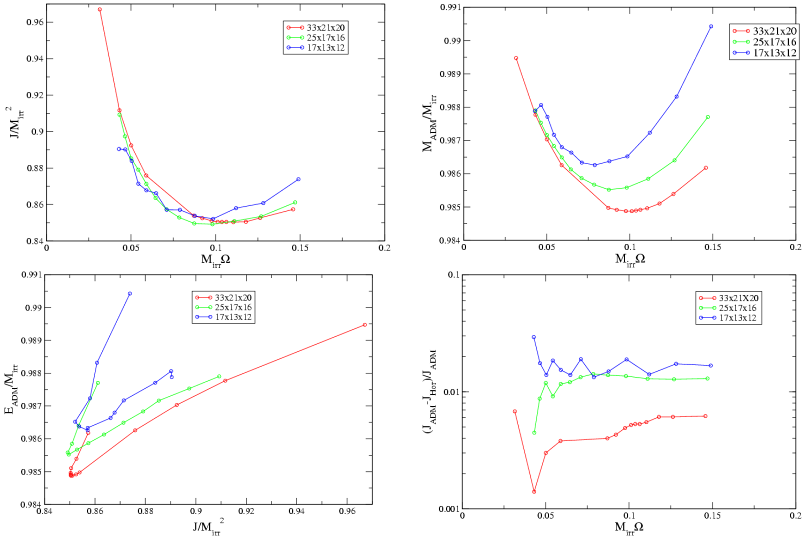
- Construction of initial data for binary black holes in quasi-circular orbits
  - CTS initial data
  - + Dynamical equation in a *waveless approximation*

- Evolution of (one) black hole

$$\Longrightarrow \begin{cases} \text{dynamical/future trapped horizons} \\ \text{fully-constrained evolution scheme} \end{cases}$$

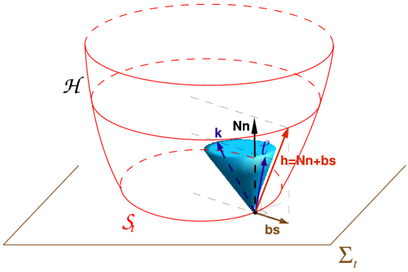
  - *Different* intuition for space-like horizons (e.g. uniqueness of the foliation...)
  - Need to study the analytical properties of the fully-constrained scheme decomposition
  - Study of the *characteristics* of the system for addressing the freedom of imposing boundary conditions on the radiative modes

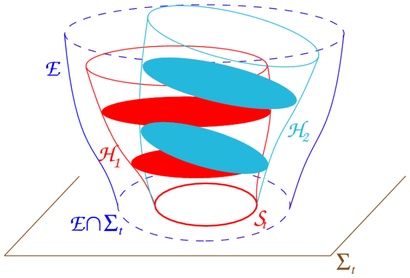
Initial data of binary black holes (F. Limousin)



Geometry of a dynamical horizon : need of new intuition...

$\boldsymbol{h} = N\boldsymbol{n} + b\boldsymbol{s}$	evolution vector on $\mathcal{H}$ associated with Σ_t
$b - N \geq 0$	elliptic equation on $b - N$
$\theta_{(k)} < 0$	are increase law
ω_μ	not an intrinsic object...
$h \to h' = \alpha h$	no rescaling invariance





Uniqueness $\mathcal{H}$-foliation theorems :

$$\boxed{\text{No unique evolution of a trapped surface } \mathcal{S}_t \text{ : dependence on } \{\Sigma_t\} \Leftrightarrow N}$$

An *optimal* dynamical horizon $\mathcal{H}$?
(maximazing area growth rate... ?)

Conclusions

- Derivation, from the Isolated Horizon formalism, of a set of boundary conditions to be imposed on a *excised* sphere, representing the quasi-equilibrium horizon of a black hole inside a generically dynamical space-time.
- Analytical translation of the BC to the CTS-like decomposition.
- Study of the interplay among geometrical, analytical, numerical (and astrophysical) requirements in the determination of such boundary conditions.
- Numerical implementation of the boundary conditions by employing spectral methods.
- Assessment of the well-posedness of the $\theta_{(\ell)} = 0 = \sigma_{(\ell)}, b = N, \kappa = \mathrm{const}$ set.
- Understanding of $\theta_{(k)}$ as the good physical parameter to prescribe on the horizon (and not its conformal transformation) : relevance for the dynamically evolving case.