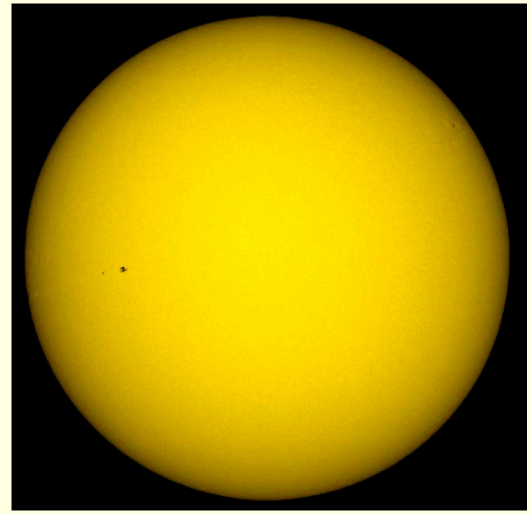
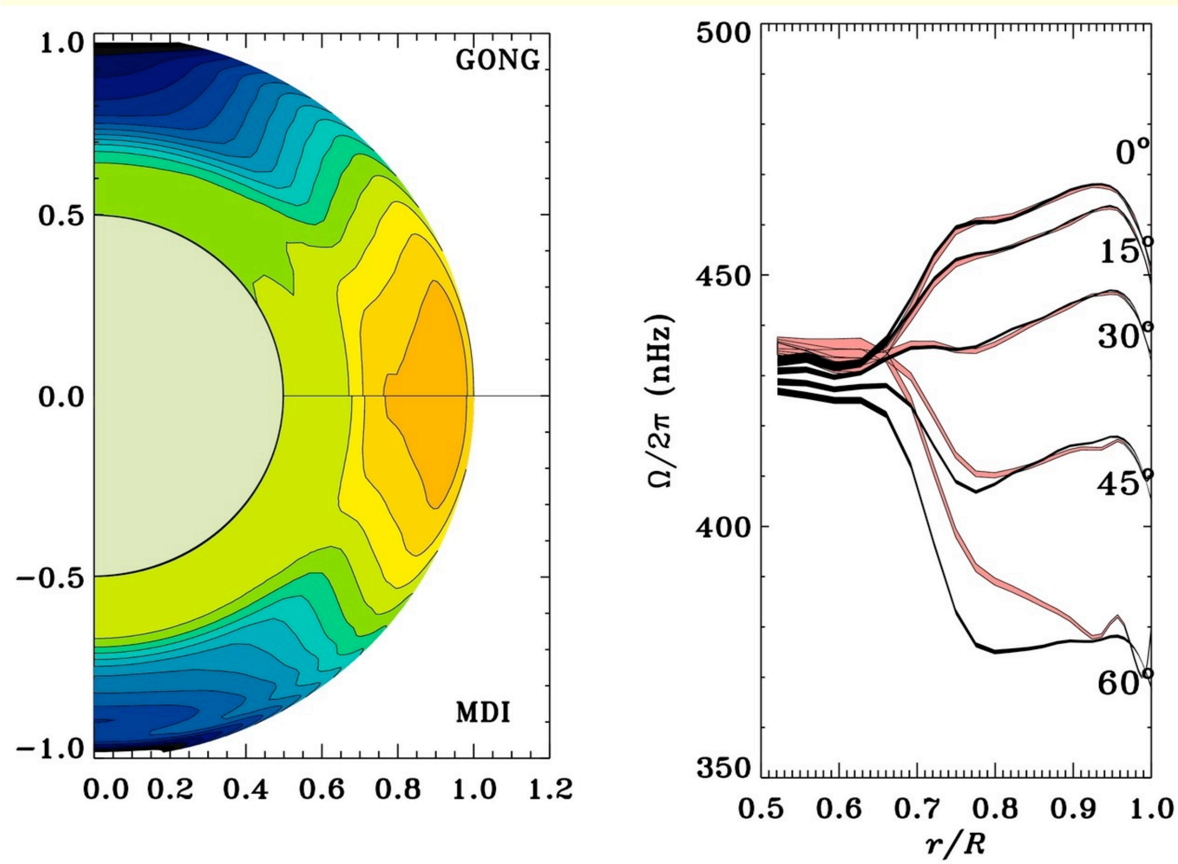
A Simple Model for the Solar Isorotation Countours





Steven Balbus
Henrik Latter
Julius Bonart
Ecole Normale Supérieure
Physics Department
Paris, France
IAP 15 jan 2009

SOLAR DIFFERENTIAL ROTATION

- One of the most beautiful astronomical results of the last half century was the precision determination of the interior solar rotation.

Splitting of p-mode frequencies allows an accurate determination of the angular velocity $\Omega(r, \theta)$, using sophisticated inversion techniques applied to the excited mode spectrum.

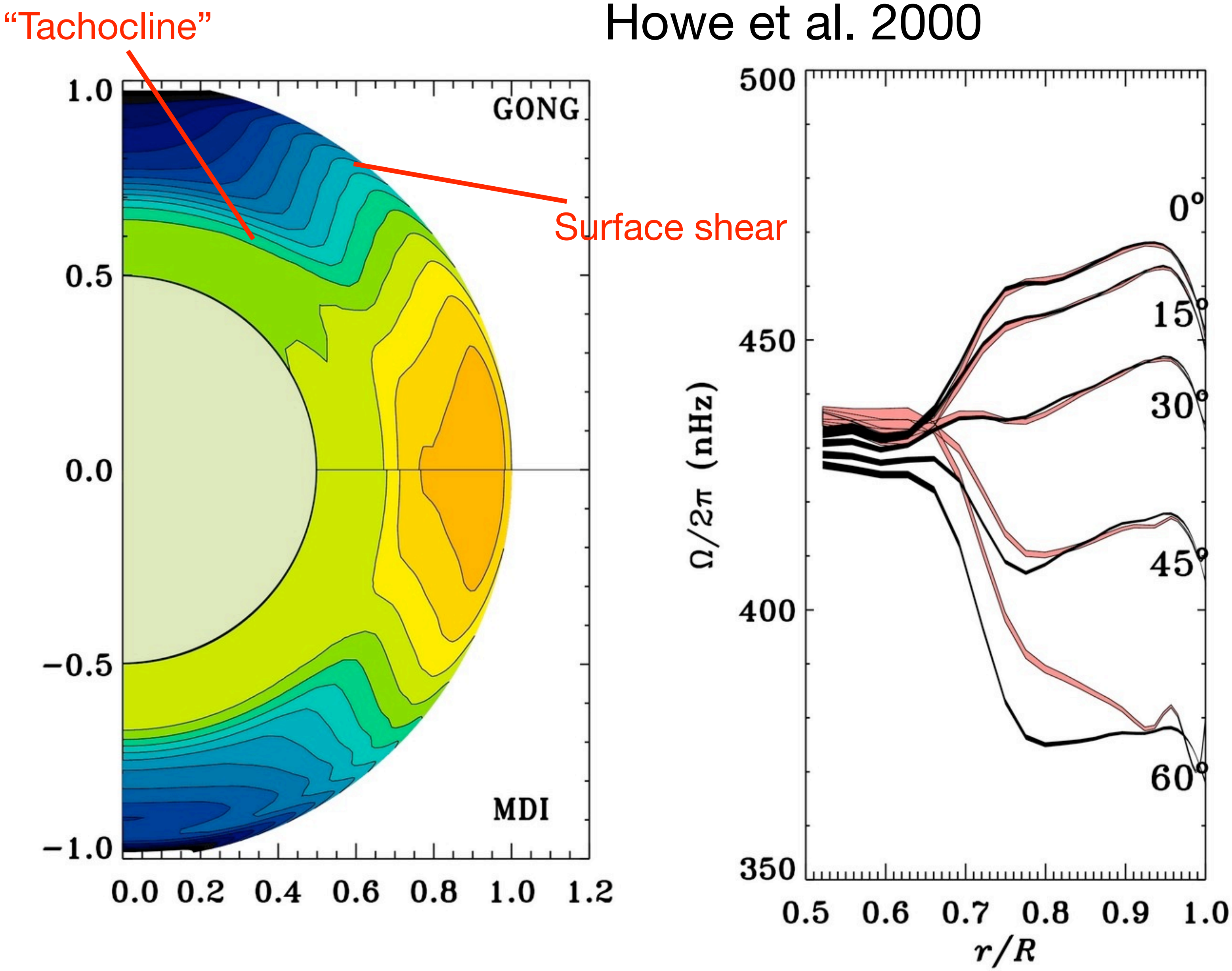
THE FINDINGS:

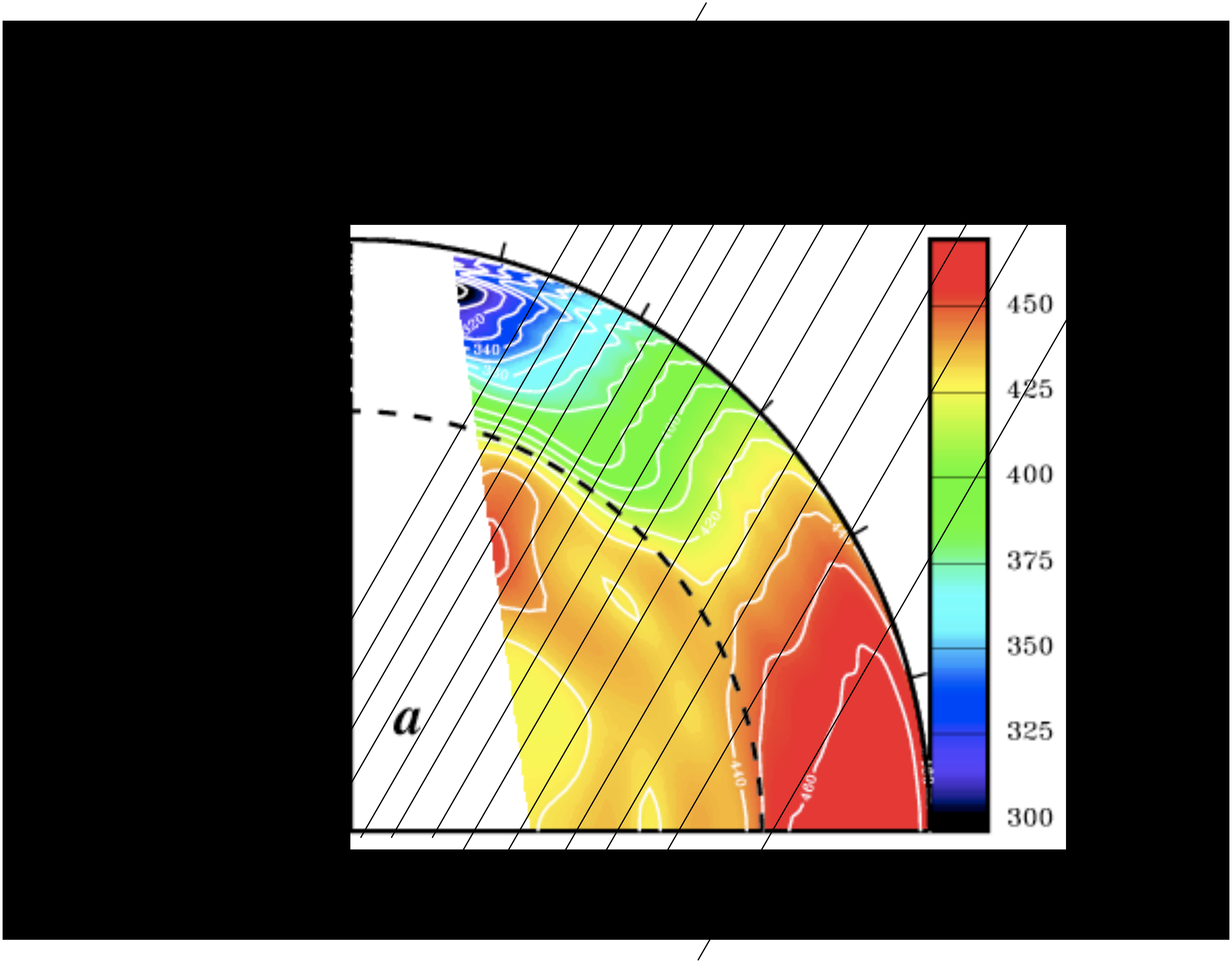
• The only place where there is significant differential rotation in the sun is in the convective zone (CZ).

• This is thought to be the only place where there is a significant level of turbulence. (So much for enhanced viscosity models.)

• The rotation is approximately constant on cones of constant θ at mid latitudes, cylindrical near the equator, spherical (apparently) near the poles.

Howe et al. 2000




a
320
340
420
440
460
450
425
400
375
350
325
300

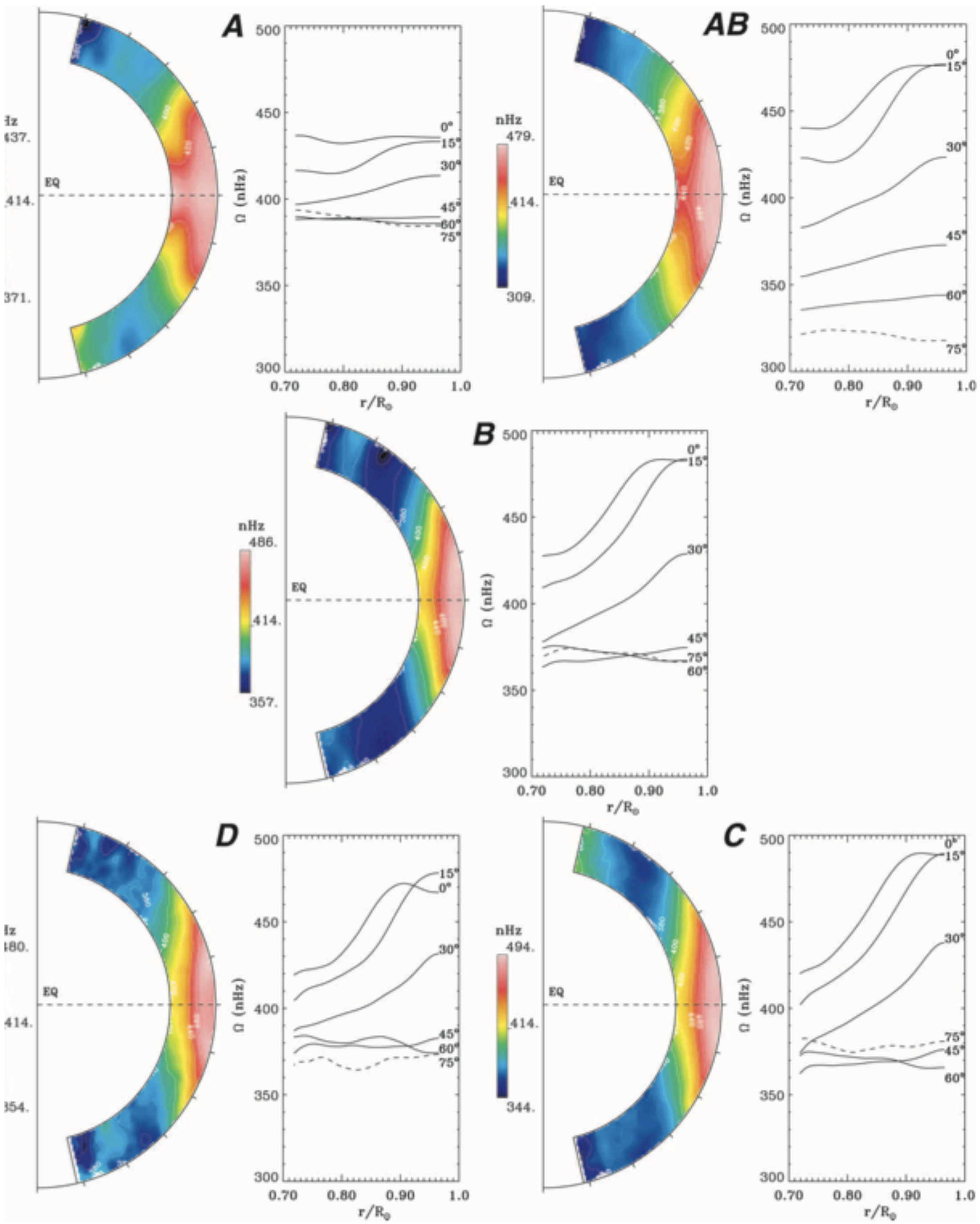
THE PROBLEM:

• The CZ is very nearly adiabatic, $P=P(\rho)$, barotropic.

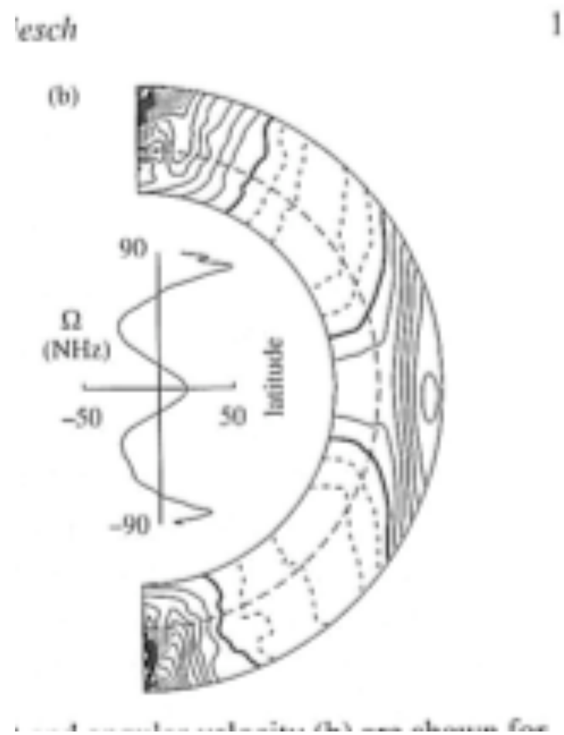
• Convective motions, except near the surface, are small… typically 30 m s^{-1}.

• A barotropic fluid in hydrostatic equilibrium must rotate on cylinders, Ω (R). (Taylor columns.) The solar rotation profile is decidedly not constant on cylinders.

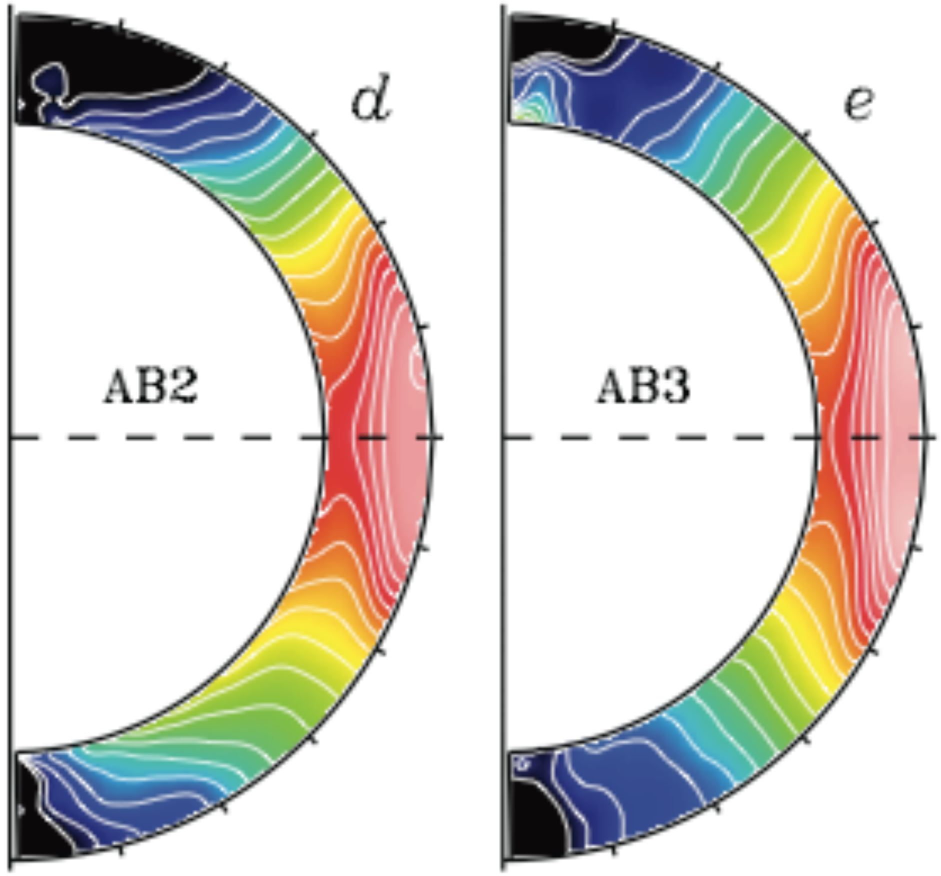
• But large scale numerical simulations generally *do* produce cylindrical contours, unless forced at boundaries.



Brun & Toomre 2002
ASH Code



Miesch, 2007



ℓ = 2 ℓ = 4

Miesch, Brun, & Toomre 2006 (imposed latitudinal ∇S.)

THE ORTHODOX VIEW

• Despite the simple regularity of the rotation pattern, the flow is an extremely complex interplay between convective turbulence and rotation. Some handles exist, however.

• Departures from barotropic structure because Coriolis forces affect convection.

• Convection along the axis of rotation is more efficient than convection in planes of constant Z. Hot poles, cool equator.

• Thermal wind equation: $R\, \partial\Omega^2/\partial z = \mathbf{e}_\varphi \cdot (\nabla P \times \nabla\rho)/\rho^2$

Cylindrical: (R, ϕ, z) spherical: (r, θ, φ)

THERMAL WIND EQUATION

$R\, \partial\Omega^2/\partial z = \mathbf{e}_\varphi \cdot (\nabla P \times \nabla\rho)/\, \rho^2$; (R, ϕ, z) or (r, θ, ϕ)

$R\, \rho^2\, \partial\Omega^2/\partial z = (\partial\rho/r\partial\theta)\, (\partial P/\partial r) - (\partial\rho/\partial r)\, (\partial P/r\partial\theta)$

Let $S = k/(\gamma\text{-}1)\ \ln P\rho^{-\gamma}$, $C_P = \gamma k/(\gamma\text{-}1)$,

$R\, \rho\, C_P\ \ \partial\Omega^2/\partial z = (\partial P/r\partial\theta)\, (\partial S/\partial r) - (\partial P/\partial r)\, (\partial S/r\partial\theta)$

For SCZ: $RC_P\, \partial\Omega^2/\partial z = g\, (\partial S/r\partial\theta)$, $\rho g = -\, (\partial P/\partial r)$.

Shows relationship between large scale latitudinal entropy gradients due to Coriolis, and departures from cylindrical "isotachs."   Trend: moving polewards, Ω *dec., S inc.*

GETTING THE LAY OF THE LAND

$N^2 = \left| g/\gamma \ \partial (\ln P\rho^{-\gamma}) / \partial r \right| \approx 3.8 \times 10^{-13} \ s^{-2}$,

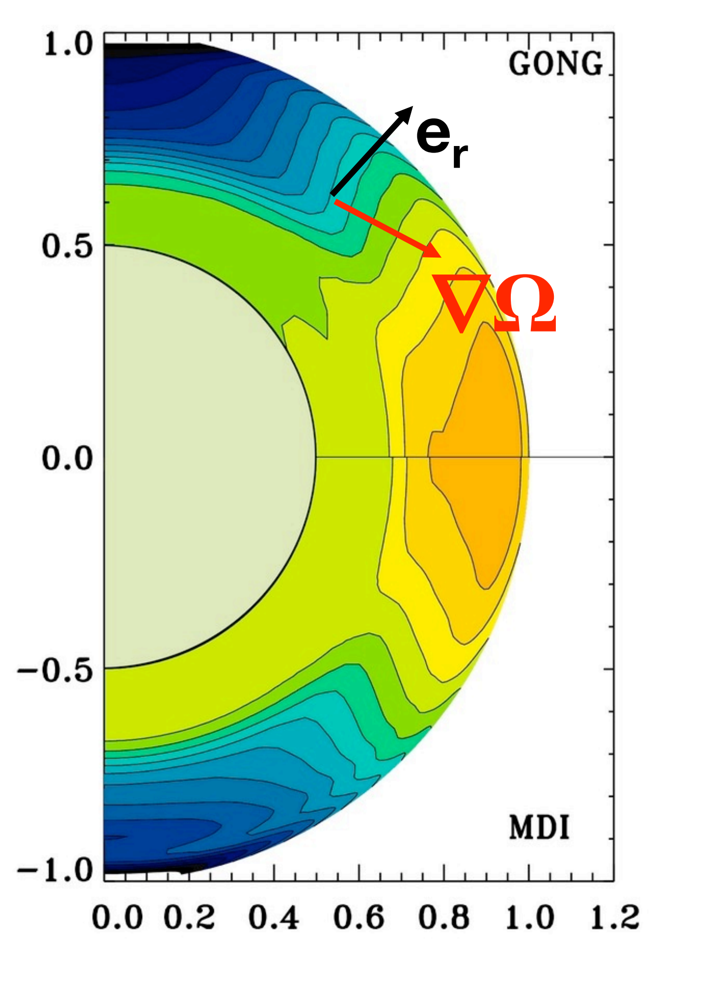
by requiring the solar luminosity to be carried by convection (Schwarzschild 1958), very rough!

But θ gradient of S is estimated by different TWE physics…

$g/\gamma \ \partial (\ln P\rho^{-\gamma}) / r\partial\theta = R\partial\Omega^2/\partial z \approx 2 \times 10^{-12} \ s^{-2}$,

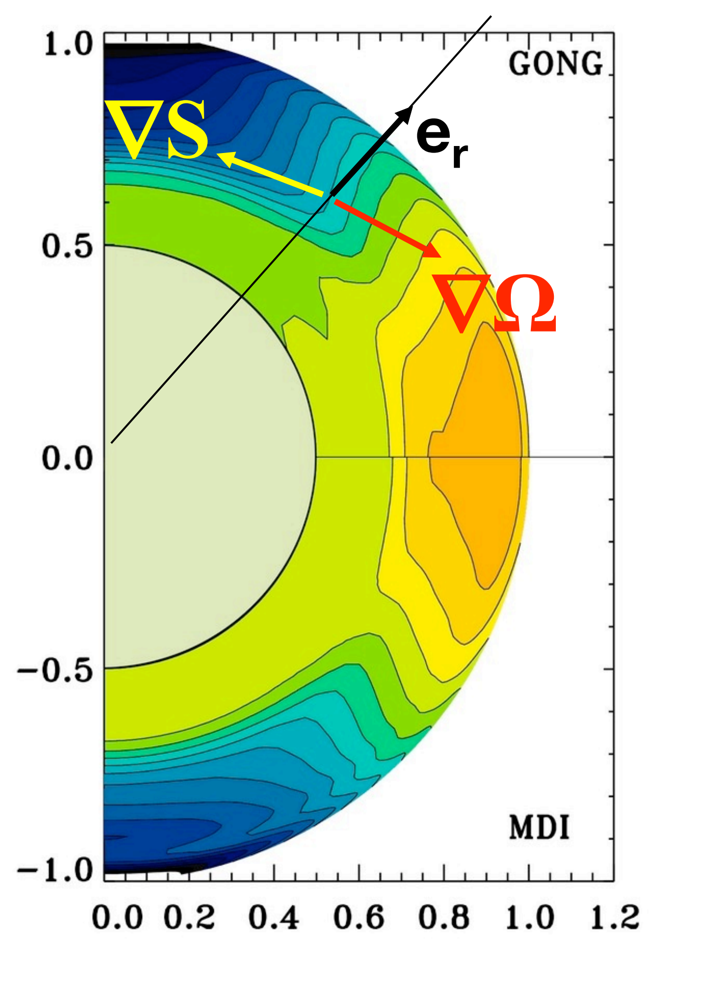
The θ gradient of S exceeds the r gradient by factor of ~ 5…if N^2 OK, and thermal wind balance valid.

∇S, $\nabla\Omega$ COUNTER ALIGNED ?



Clearly, $\mathbf{e}_\theta \bullet \nabla\Omega$ also much exceeds $\mathbf{e}_r \bullet \nabla\Omega$.

∇S, $\nabla\Omega$ COUNTER ALIGNED ?



Clearly, $\mathbf{e}_\theta \bullet \nabla\Omega$ also much exceeds $\mathbf{e}_r \bullet \nabla\Omega$.

What if $\nabla\Omega$ and ∇S are more closely related than just a trend? What if $S=S(\Omega^2)$?

TWE would then define the isorotational surfaces.

Thermal Wind Equation with S=S(Ω^2):

$$\frac{\partial \Omega^2}{\partial r} - \left(\frac{gS'}{C_P \sin\theta \cos\theta} + \frac{\tan\theta}{r} \right) \frac{\partial \Omega^2}{\partial \theta} = 0.$$

where S' is dS/dΩ^2. Solution is Ω^2 is constant along the characteristic

$$\frac{d\theta}{dr} = - \left(\frac{gS'}{C_P r^2 \sin\theta \, \cos\theta} + \frac{\tan\theta}{r} \right)$$

Since Ω is constant along this characteristic, so is S'.
To solve, set y = sin θ. Find:

$$\frac{dy^2}{dr} + \frac{2y^2}{r} = -\frac{2gS'}{C_P r^2},$$ a first order linear equation.

With $g = GM_\odot/r^2$, the solution is:

$$r^2 \sin^2\theta = R^2 = A - \frac{B}{r}$$

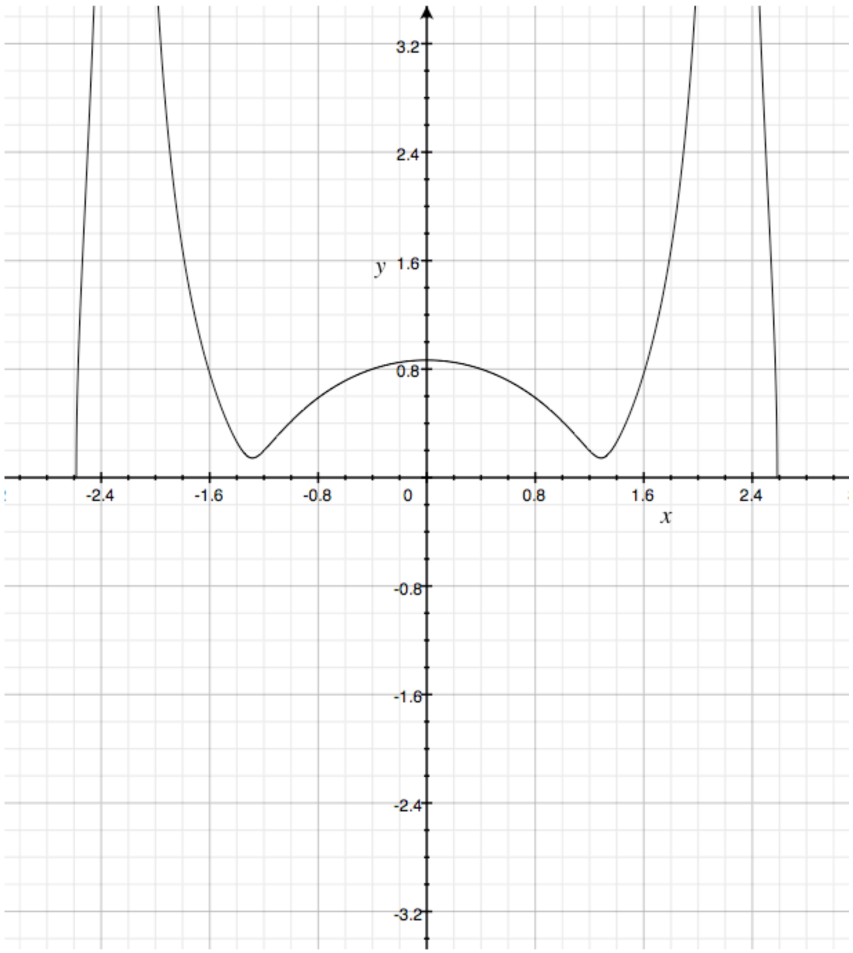
where A is an integration constant, and B is

$$B = -\frac{2GM_\odot S'}{C_P}$$

$B/r_\odot^3 \sim$ order unity or less.

The basic result is clear:

R small, r=const.
$r \gg B/2A$, R=const.



"Batman contour" is typical. Spheres at small R, cylinders at larger R, sharp upturn in between.

$$y = \left[\frac{\alpha^2}{(1 - \beta x^2)^2} - x^2 \right]^{1/2}$$

With $g = GM_\odot/r^2$, the solution is:

$$r^2 \sin^2\theta = R^2 = A - \frac{B}{r}$$

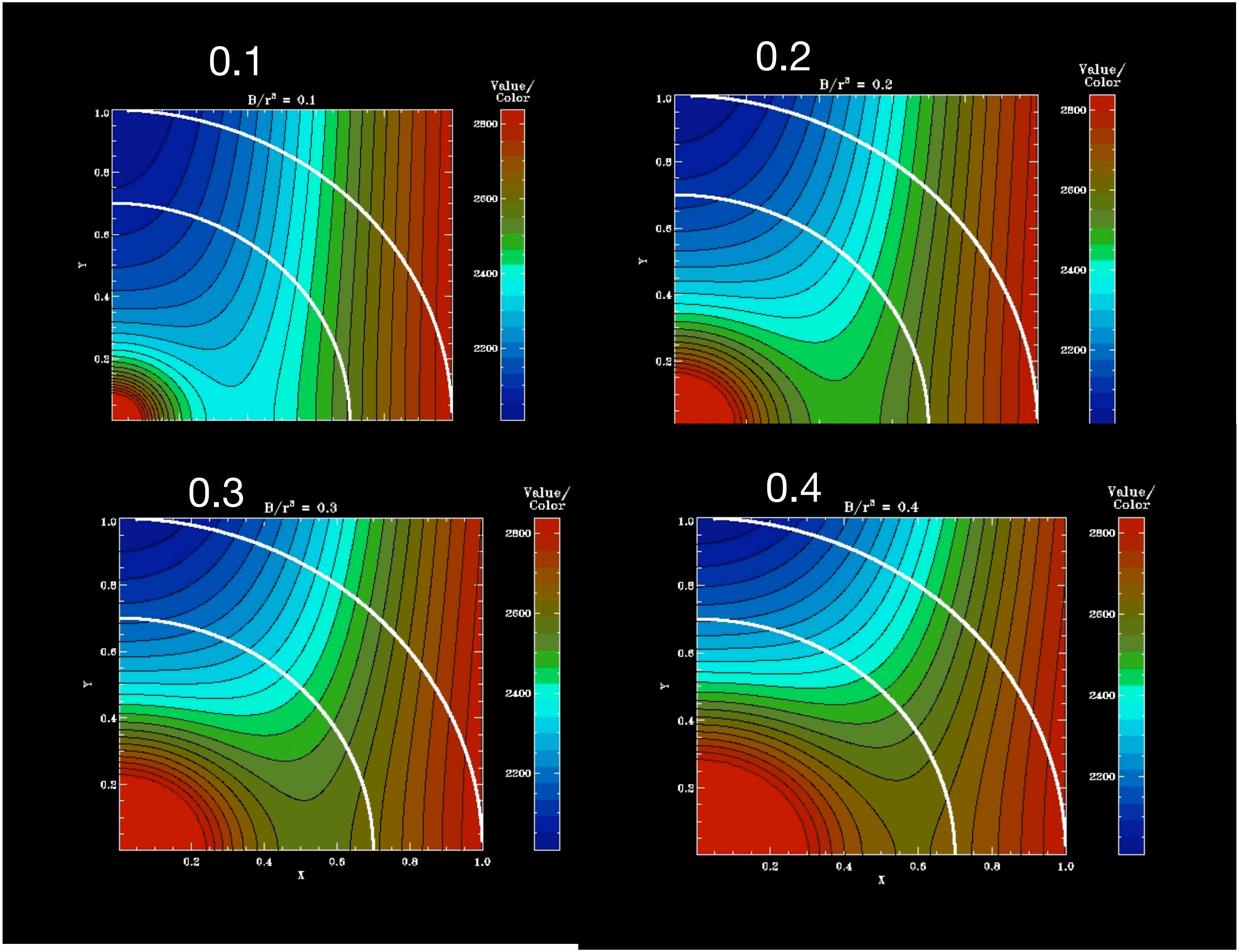
where A is an integration constant, and B is

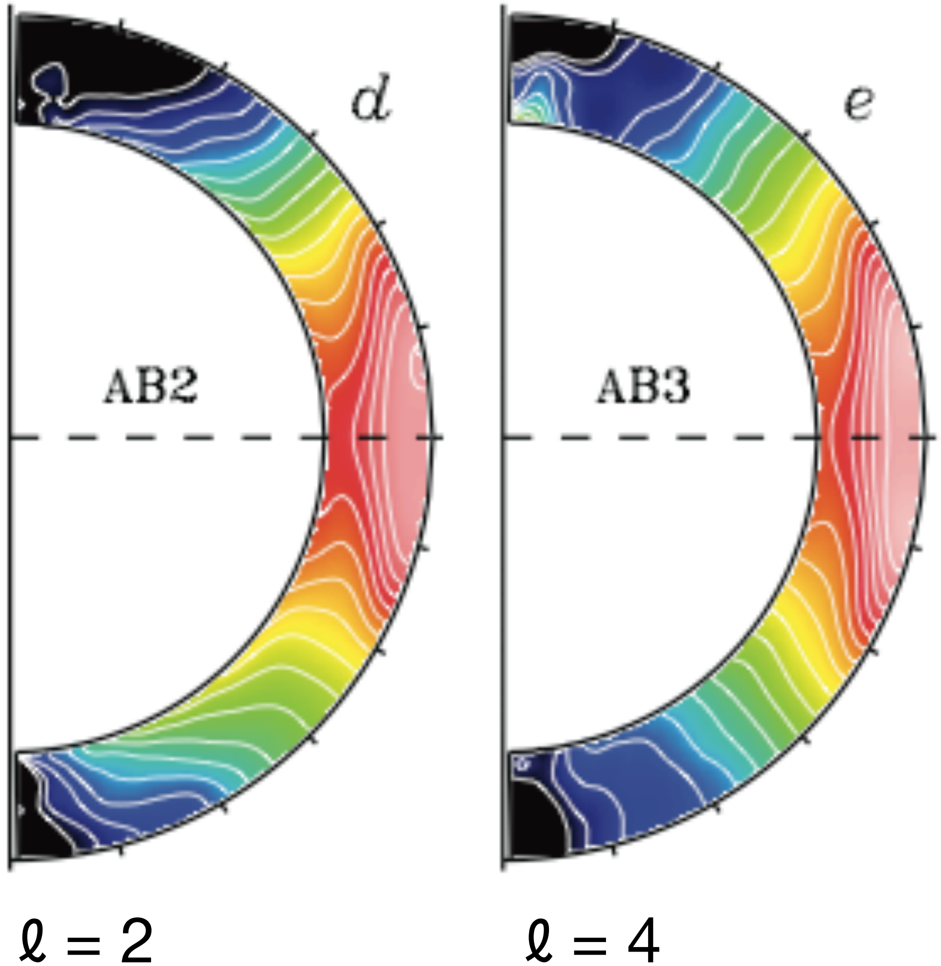
$$B = -\frac{2GM_\odot S'}{C_P}$$

$B/r_\odot^3$ ~ order unity or less.

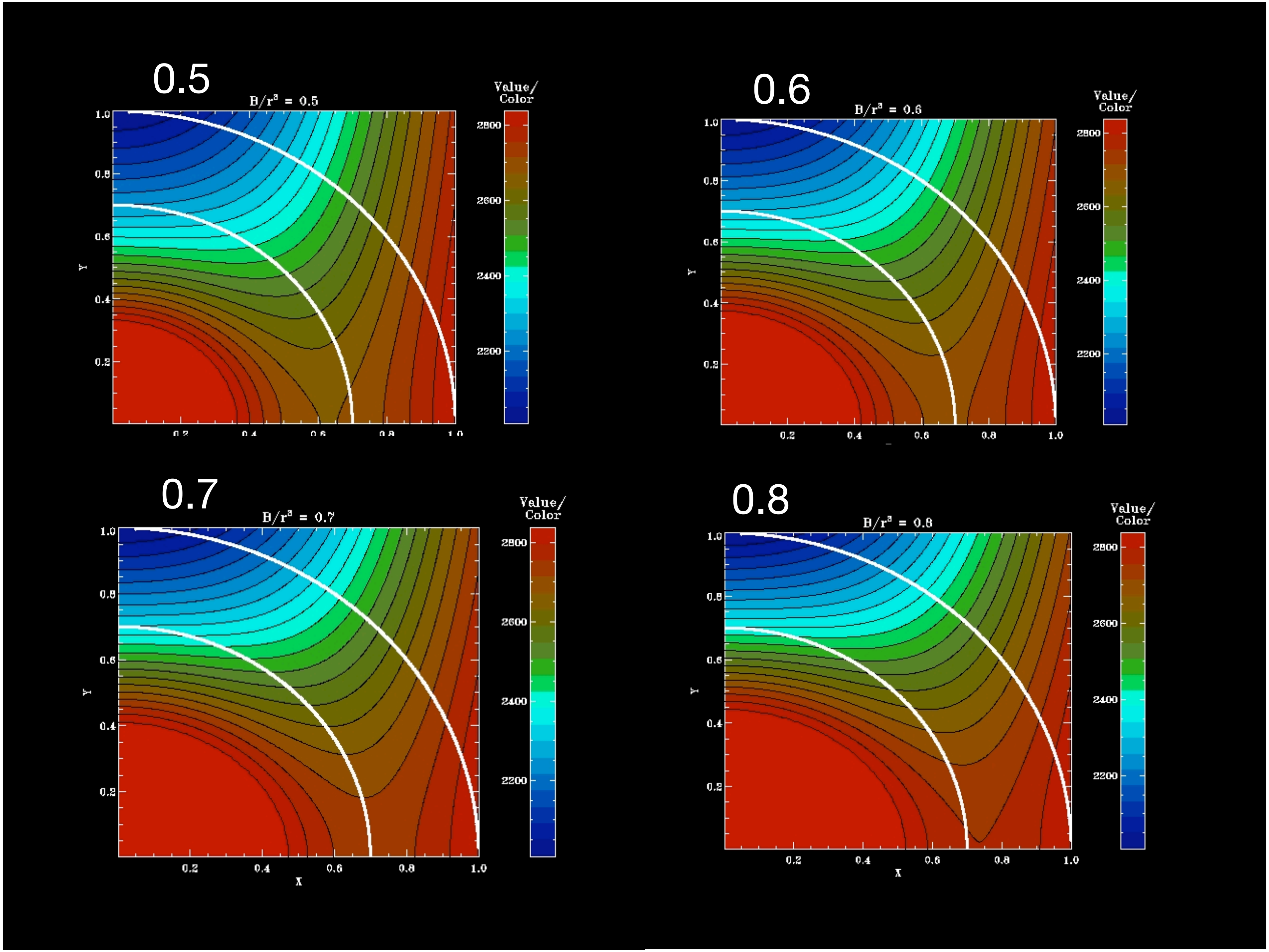
For solving for Ω, assume a fit at $r=r_\odot$, $\Omega(\cos^2\theta_o)$, where θ_o is θ at $r=r_\odot$, the starting point of the characteristic

$$r^2 \sin^2\theta = r_\odot^2 \sin^2\theta_0 + B\left(\frac{1}{r_\odot} - \frac{1}{r}\right)$$


0.1
B/r³ = 0.1
0.2
B/r³ = 0.2
0.3
B/r³ = 0.3
0.4
B/r³ = 0.4
Value/ Color
2800
2600
2400
2200
X
Y
0.2
0.4
0.6
0.8
1.0



Miesch, Brun, & Toomre 2006 (imposed latitudinal ∇.)


0.5
B/r³ = 0.5
Value/ Color
0.6
B/r³ = 0.6
Value/ Color
0.7
B/r³ = 0.7
Value/ Color
0.8
B/r³ = 0.8
Value/ Color
X
Y
2800
2600
2400
2200
0.2
0.4
0.6
0.8
1.0

Thermal Wind Equation for $S=S(L^2)$:

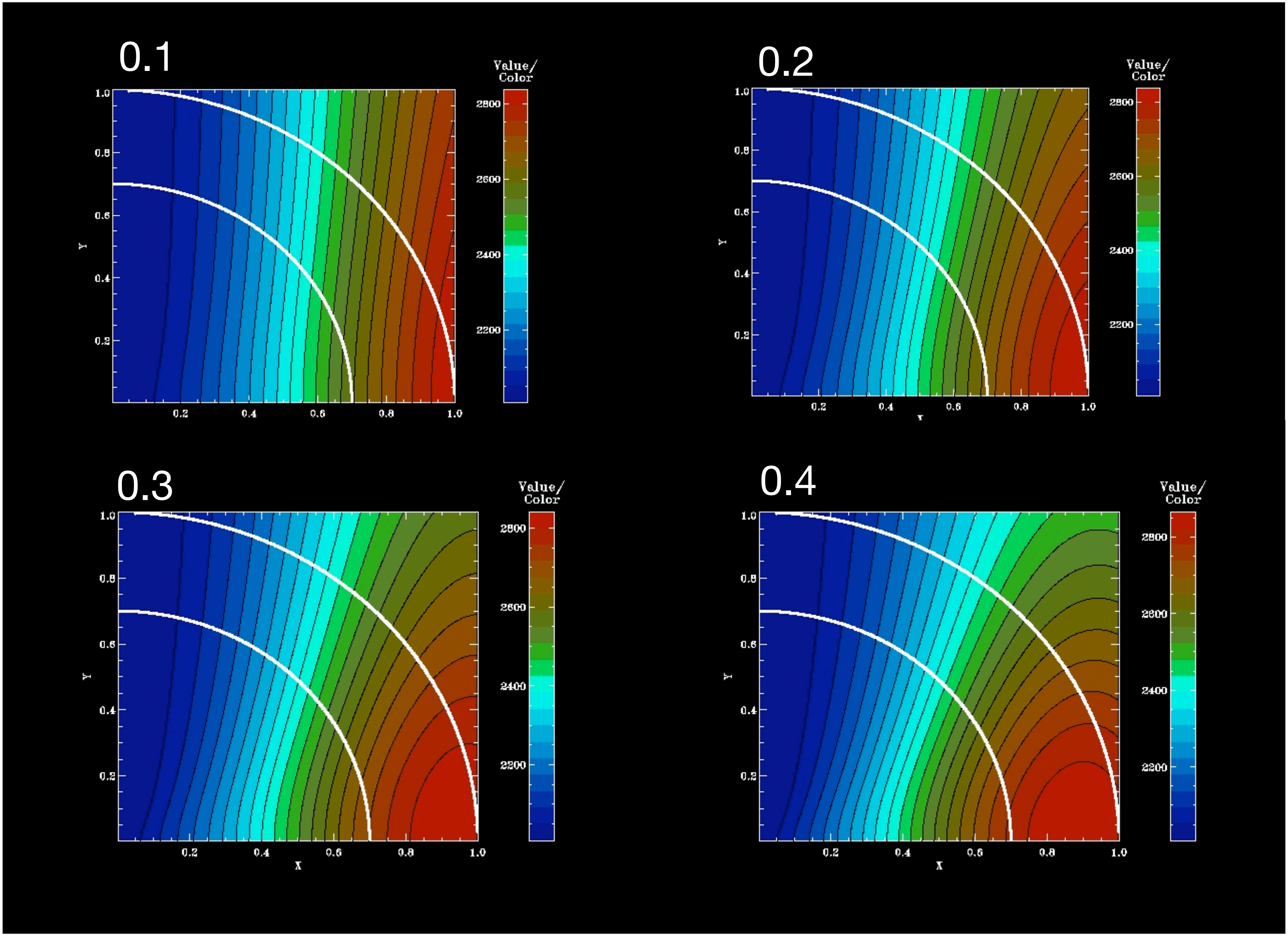
$$\frac{\partial \ell^2}{\partial r} - \tan\theta \left(\frac{1}{r} + \frac{g r^2 \sin^2\theta S_{\ell^2}}{C_P} \right) \frac{\partial \ell^2}{\partial \theta} = 0, \quad S_{\ell^2} = dS/d\ell^2$$

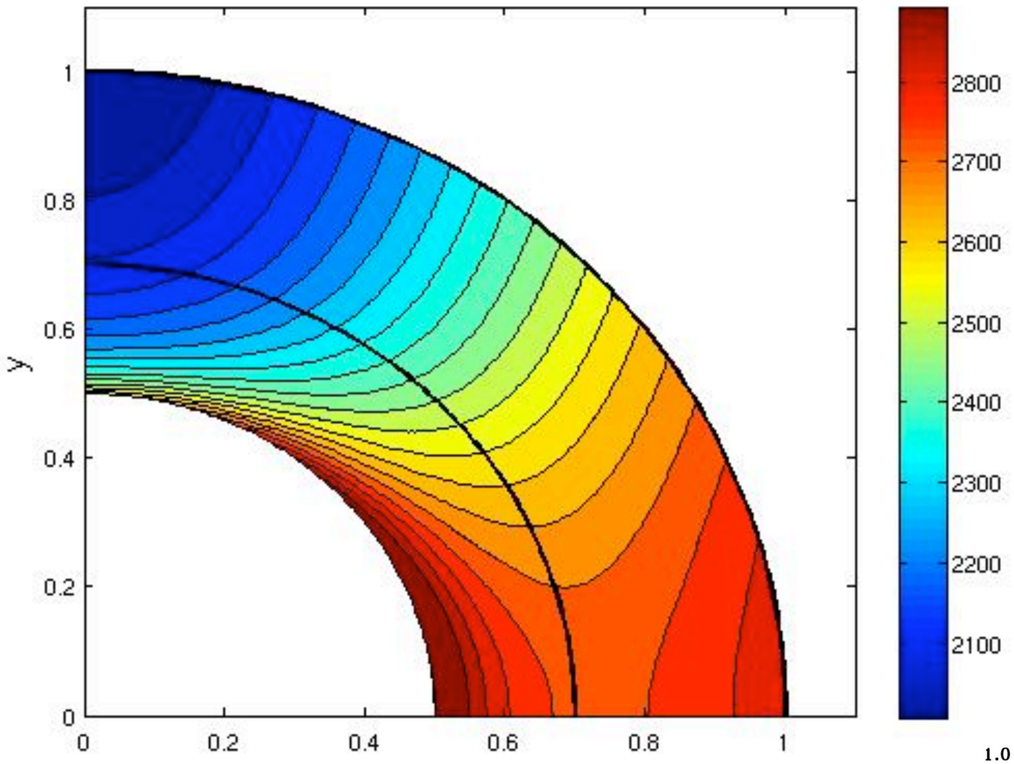
Solution is angular momentum is const. along characteristic

$$\frac{d\theta}{dr} = -\tan\theta \left(\frac{1}{r} + \frac{g r^2 \sin^2\theta S_{\ell^2}}{C_P} \right)$$

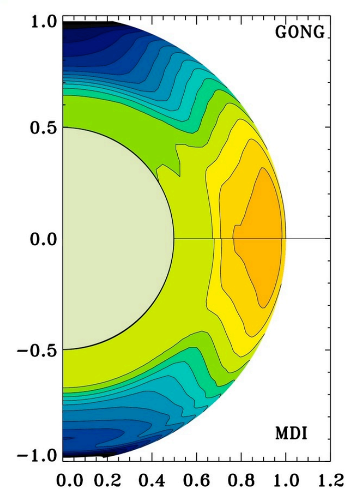
Solution is similar to angular velocity characteristics.
Find:

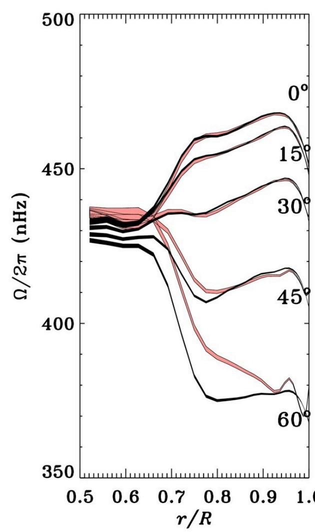
$$\frac{1}{R^2} = \frac{1}{r^2 \sin^2\theta} = A + \frac{B}{r}, \quad B = -\frac{2GM_\odot S_{\ell^2}}{C_P}$$

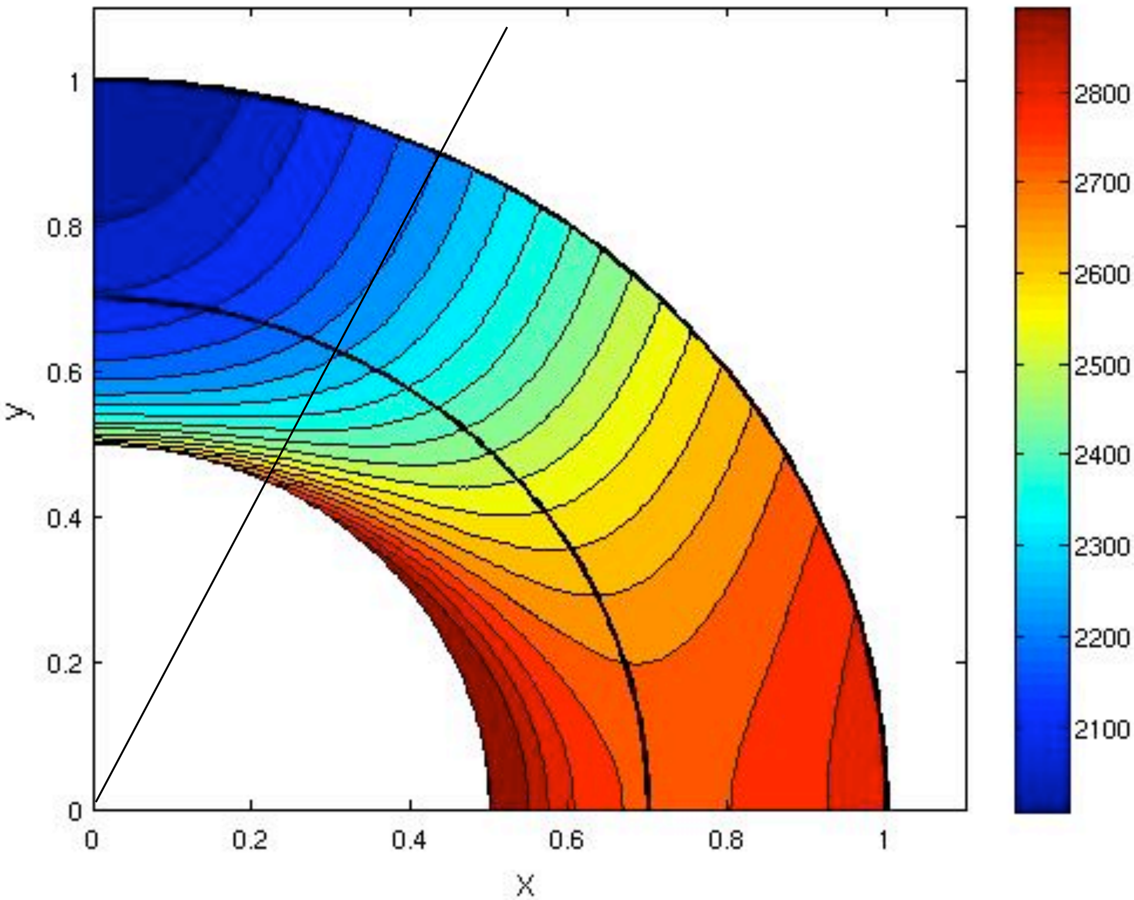

0.1
0.2
0.3
0.4
Value/
Color
2800
2600
2400
2200
0.2
0.4
0.6
0.8
1.0
X
Y



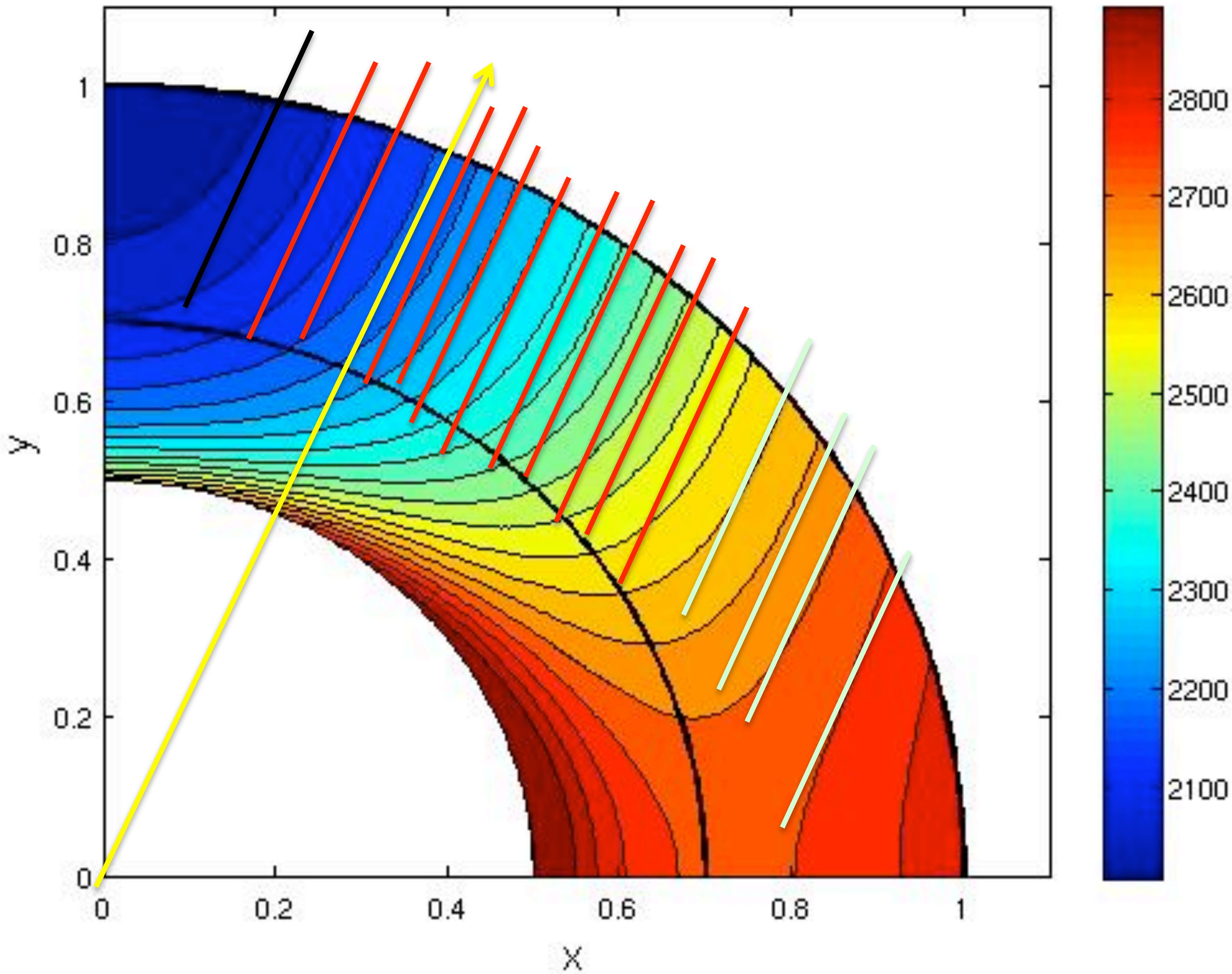
$$B/r_{\odot}^3 = 0.12 + 0.8\sin^2\theta_0$$



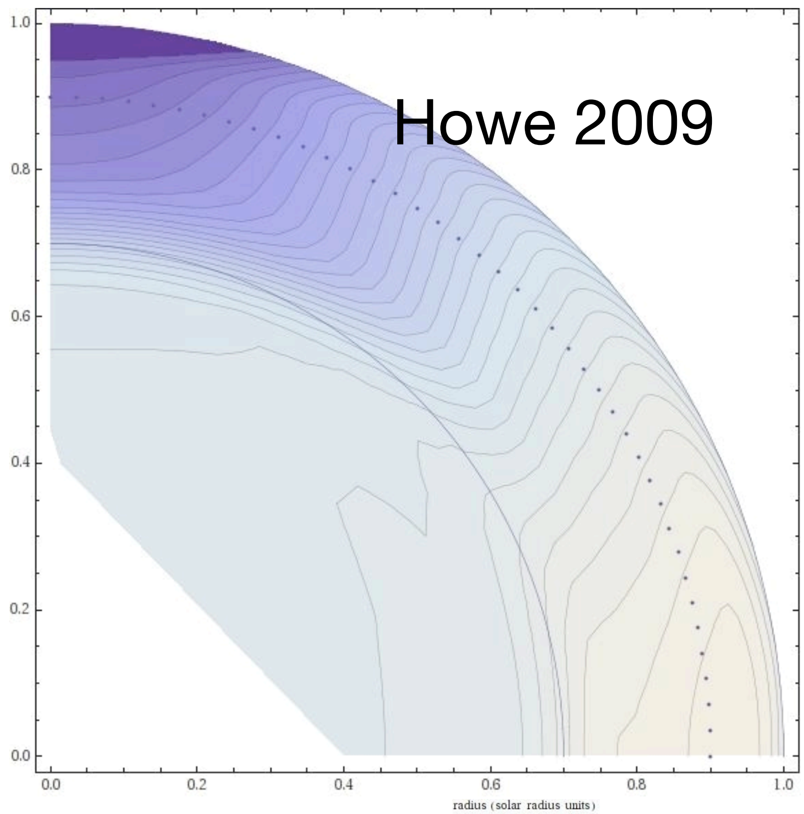


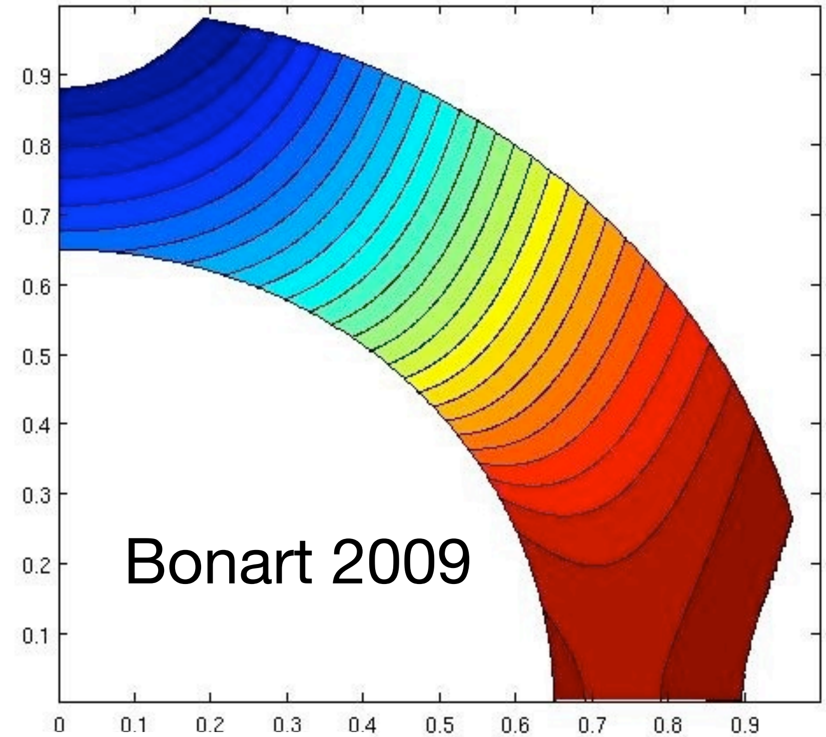


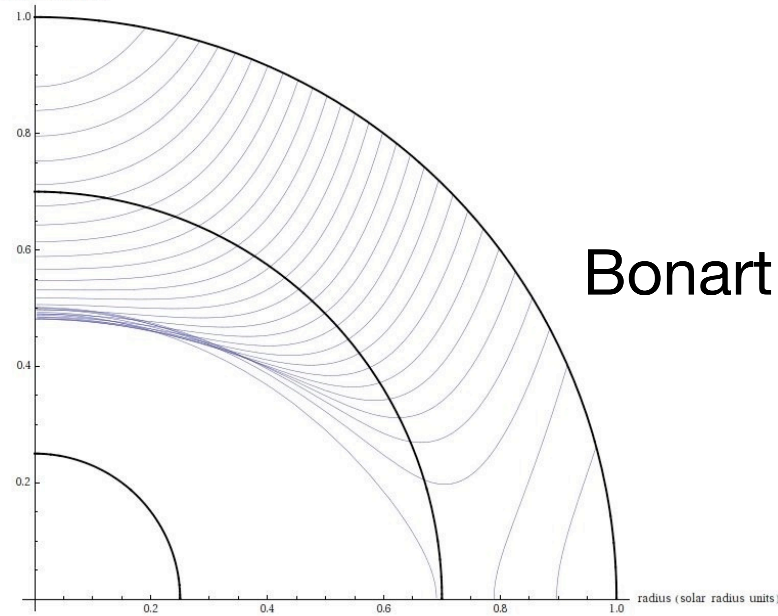
$$B/r_{\odot}^{3} = 0.12 + 0.8\sin^{2}\theta_{0}$$



1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x
2800
2700
2600
2500
2400
2300
2200
2100


Howe 2009
radius (solar radius units)


Bonart 2009


Bonart 2009
radius (solar radius units)

This is often the way it is in physics---our mistake is not that we take our theories too seriously, but that we do not take them seriously enough.

---Steven Weinberg, in The First Three Minutes

HOW IS IT THAT S AND Ω CARE ABOUT EACH OTHER SO MUCH?

To answer this, we need to understand something about the stability of rotating, stratified, magnetized plasmas.

We need to take rotation, stratification and magnetism seriously.

THE PUNCHLINE:

Counter alignment of the entropy and angular velocity gradients is a rigorous condition for marginal stability in a rotating, convective, magnetized gas.

THE PUNCHLINE:

The solar rotation profile can be understood as a consequence of maintaining a state of marginal (in)stability to the most rapidly growing axi- and nonaxisymmetric dynamical modes.

THE PUNCHLINE:

A magnetic field is essential to this picture.

Fundamental linear response of a magnetized medium:

$$(\omega^2 - (\mathbf{k} \cdot \mathbf{v_A})^{\mathbf{2}})^{\mathbf{2}} = \mathbf{0}$$

$$\omega^4 - 2\omega^2 (\mathbf{k} \cdot \mathbf{v_A})^{\mathbf{2}} + (\mathbf{k} \cdot \mathbf{v_A})^{\mathbf{4}} = \mathbf{0}$$

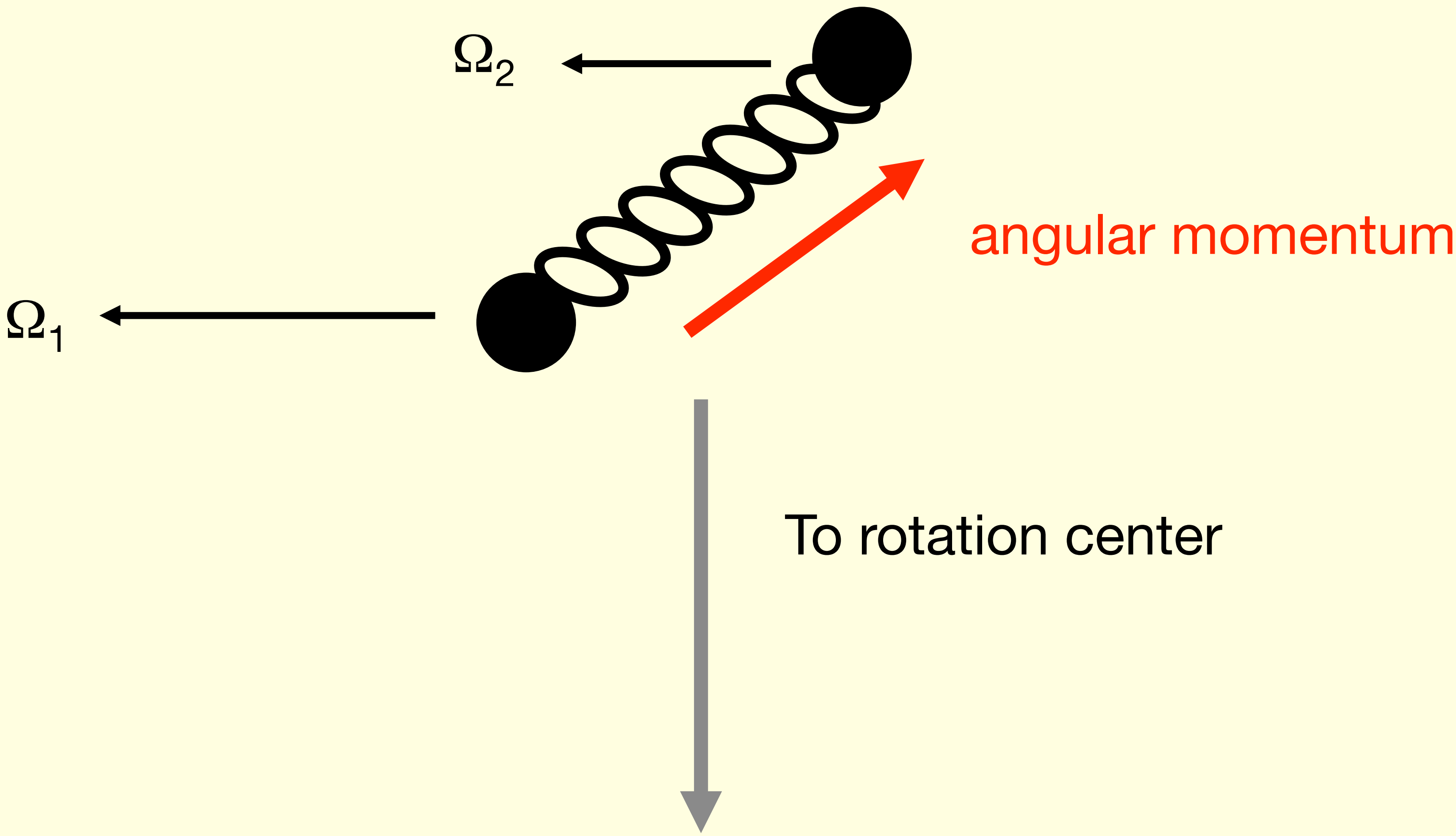
(Boussinesq; degenerate Alfvén & slow modes.)

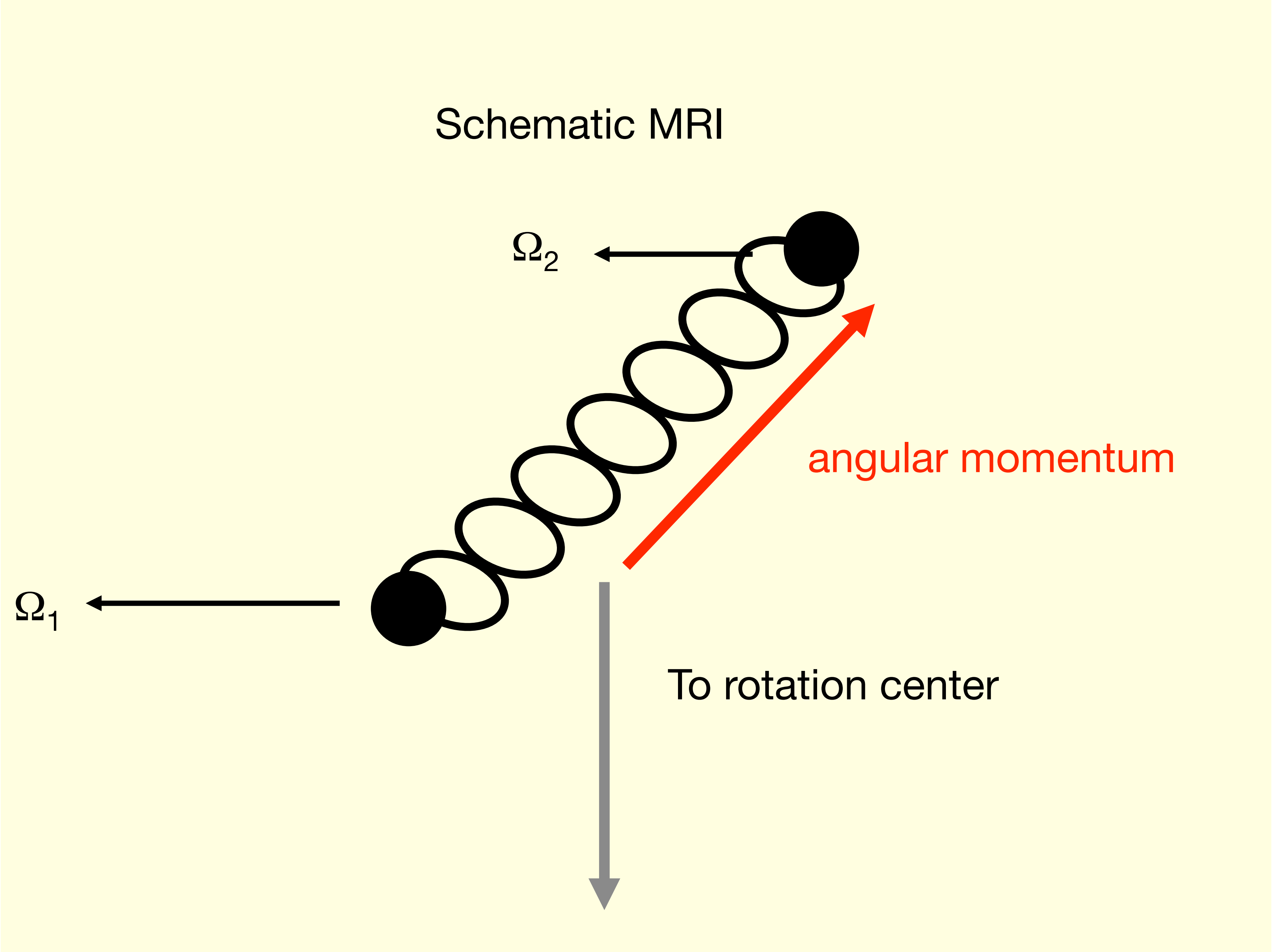
Addition of rotation introduces two new terms,

$$\omega^4 - \omega^2 \boxed{(\kappa^2} + 2(\mathbf{k} \cdot \mathbf{v_A})^{\mathbf{2}}) + (\mathbf{k} \cdot \mathbf{v_A})^{\mathbf{4}} + \boxed{(\mathbf{k} \cdot \mathbf{v_A})^{\mathbf{2}} \frac{\mathbf{d\Omega^2}}{\mathbf{d \ln R}}} = \mathbf{0}$$

one of which is “epicyclic,” $\kappa^2 = d\Omega^2/d\ln R + 4\Omega^2$,
the other of which is “tethering,” and gives rise to the MRI.

Schematic MRI




Schematic MRI
Ω2
angular momentum
Ω1
To rotation center

Compact form of equation:

$$\varpi^4 - \varpi^2\kappa^2 - 4\Omega^2(\mathbf{k}\cdot\mathbf{v_A})^\mathbf{2} = \mathbf{0}, \qquad \varpi^\mathbf{2} = \omega^\mathbf{2} - (\mathbf{k}\cdot\mathbf{v_A})^\mathbf{2}$$

General wave numbers:

$$\frac{k^2}{k_z^2}\varpi^4 - \varpi^2\kappa^2 - 4\Omega^2(\mathbf{k}\cdot\mathbf{v_A})^\mathbf{2} = \mathbf{0}, \qquad \varpi^\mathbf{2} = \omega^\mathbf{2} - (\mathbf{k}\cdot\mathbf{v_A})^\mathbf{2}$$

Allow Ω(R, z):

$$\frac{k^2}{k_z^2}\varpi^4 + \varpi^2\left[\frac{\mathcal{D}(R^4\Omega^2)}{R^3}\right] - 4\Omega^2(\mathbf{k}\cdot\mathbf{v_A})^\mathbf{2} = \mathbf{0},$$

$$\mathcal{D} = \left(\frac{k_R}{k_z}\frac{\partial}{\partial z} - \frac{\partial}{\partial R}\right)$$

Allow S(R,z) as well:

$$\frac{k^2}{k_z^2}\varpi^4 + \varpi^2\left[\frac{1}{\gamma\rho}(\mathcal{D}P)(\mathcal{D}\ln P\rho^{-\gamma}) + \frac{\mathcal{D}(R^4\Omega^2)}{R^3}\right] - 4\Omega^2(\mathbf{k}\cdot\mathbf{v_A})^{\mathbf{2}} = \mathbf{0},$$

Most general, baroclinic, axisymmetric response.
Stability from $\omega \rightarrow 0$ limit:

$$\frac{1}{\gamma\rho}(\mathcal{D}P)(\mathcal{D}\ln P\rho^{-\gamma}) + \frac{\mathcal{D}(R^4\Omega^2)}{R^3} + 4\Omega^2 < 0 \quad \text{for stability}$$

$$\frac{1}{\gamma\rho}(\mathcal{D}P)(\mathcal{D}\ln P\rho^{-\gamma}) + \frac{\mathcal{D}(R^4\Omega^2)}{R^3} < 0 \quad \text{for } \textit{hydro} \text{ stability}$$

More clear written in terms of displacement vector, $\xi\mathbf{n}$

$$\frac{k^2}{k_z^2}\varpi^4 - \varpi^2\left[\frac{gn_r}{\gamma}(\mathbf{n}\cdot\nabla\ln P\rho^{-\gamma}) + n_R\frac{\mathbf{n}\cdot\nabla(R^4\Omega^2)}{R^3}\right] - 4\Omega^2(\mathbf{k}\cdot\mathbf{v_A})^\mathbf{2} = \mathbf{0},$$

Then,

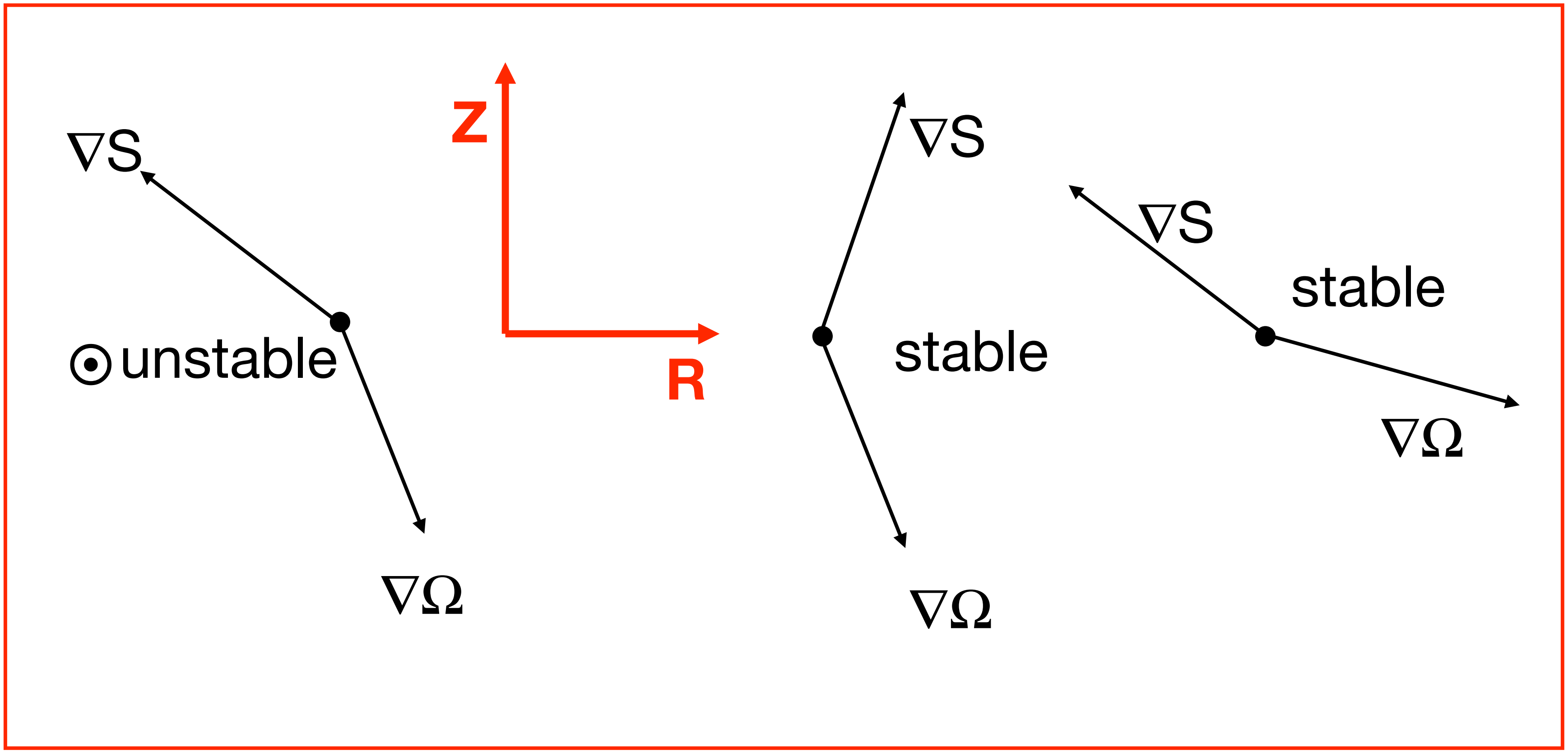
$$\frac{gn_r}{\gamma}(\mathbf{n}\cdot\nabla)\ln P\rho^{-\gamma} + Rn_R(\mathbf{n}\cdot\nabla)\Omega^2 > 0 \quad \text{for stability}$$

Marginal modes exist when rotation and entropy surfaces coincide. Explicitly (Papaloizou & Szuszkiewicz 1992, Balbus 1995):

$$\left(-\frac{\partial P}{\partial z}\right)\left(\underset{+}{\frac{\partial\Omega^2}{\partial R}}\,\underset{+}{\frac{\partial\ln P\rho^{-\gamma}}{\partial z}} - \underset{-}{\frac{\partial\Omega^2}{\partial z}}\,\underset{-}{\frac{\partial\ln P\rho^{-\gamma}}{\partial R}}\right) > 0 \quad \text{for stability}$$

$N^2 + d\Omega^2/d\ln R > 0$ also required.

$$\left(-\frac{\partial P}{\partial z}\right)\left(\frac{\partial \Omega^2}{\partial R}\frac{\partial \ln P\rho^{-\gamma}}{\partial z} - \frac{\partial \Omega^2}{\partial z}\frac{\partial \ln P\rho^{-\gamma}}{\partial R}\right) > 0 \quad \text{for stability}$$



Did we miss something? What happened to good old-fashioned convection?

$$\frac{k^2}{k_z^2}\varpi^4 + \varpi^2\left[\frac{1}{\gamma\rho}(\mathcal{D}P)(\mathcal{D}\ln P\rho^{-\gamma}) + \frac{\mathcal{D}(R^4\Omega^2)}{R^3} - \frac{m^2}{k_z^2 R^2}N^2\right] - 4\Omega^2(\mathbf{k}\cdot\mathbf{v_A})^{\mathbf{2}} = \mathbf{0},$$

$$N^2 = -\frac{1}{\gamma\rho}\frac{\partial P}{\partial r}\frac{\partial \ln P\rho^{-\gamma}}{\partial r} = \frac{g}{\gamma}\frac{\partial \ln P\rho^{-\gamma}}{\partial r}$$

BV oscillations picked out by *nonaxisymmetric* modes, in uniformly rotating medium.

Without a magnetic field, these purely hydrodynamic modes dominate the question of stability. With even a weak magnetic field, the axisymmetric modes become major players.

In nonaxisymmetry, local wavenumbers are stretched:

$$k_R(t) = k_R(0) - m\frac{\partial\Omega}{\partial R}t$$

$$k_z(t) = k_z(0) - m\frac{\partial\Omega}{\partial z}t$$

Hence, the wavenumber ratio k_R / k_Z tends toward

$$\frac{k_R}{k_z} \rightarrow \frac{\partial\Omega/\partial R}{\partial\Omega/\partial z}$$

and . . .

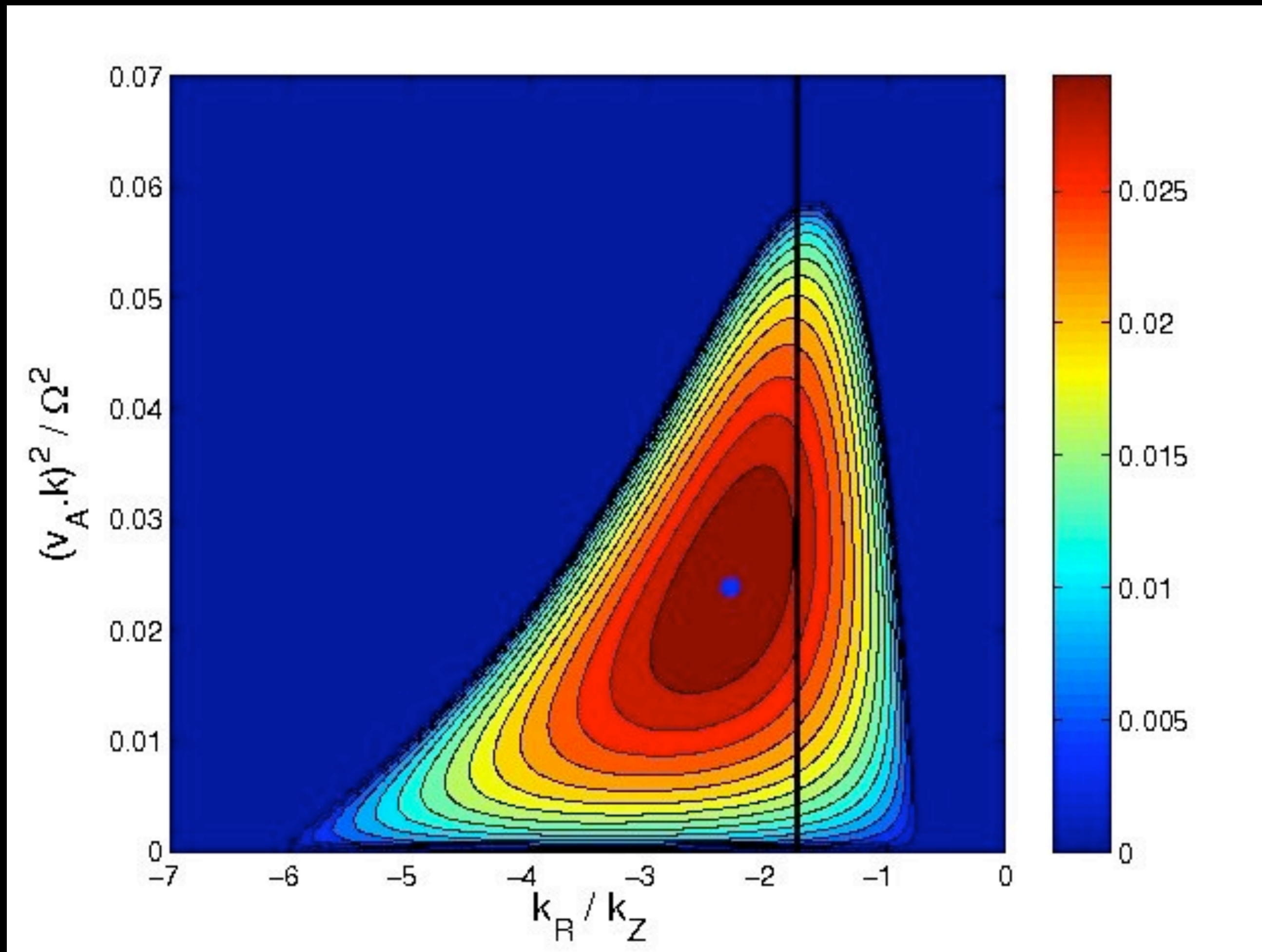
The D operator takes on an interesting form:

$$\mathcal{D}S = \left(\frac{k_R}{k_z}\frac{\partial S}{\partial z} - \frac{\partial S}{\partial R}\right) = \left(\frac{\partial \Omega}{\partial z}\right)^{-1} \nabla S \times \nabla \Omega$$

$$S = \ln P\rho^{-\gamma}$$

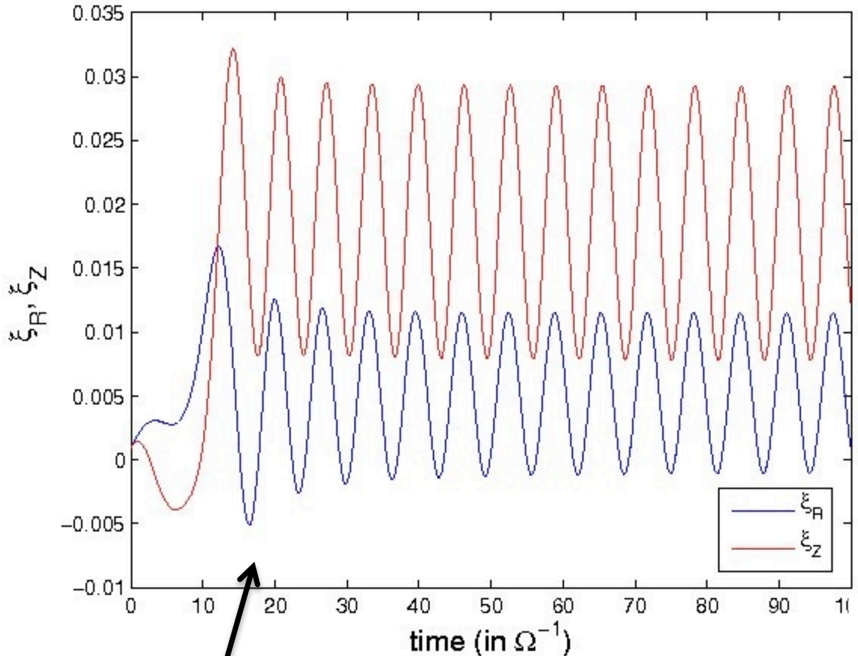
$$\mathcal{D}\Omega = \left(\frac{k_R}{k_z}\frac{\partial \Omega}{\partial z} - \frac{\partial \Omega}{\partial R}\right) = 0$$

In other words, marginalizing D---thus the dynamical response---corresponds to aligning S and Ω.

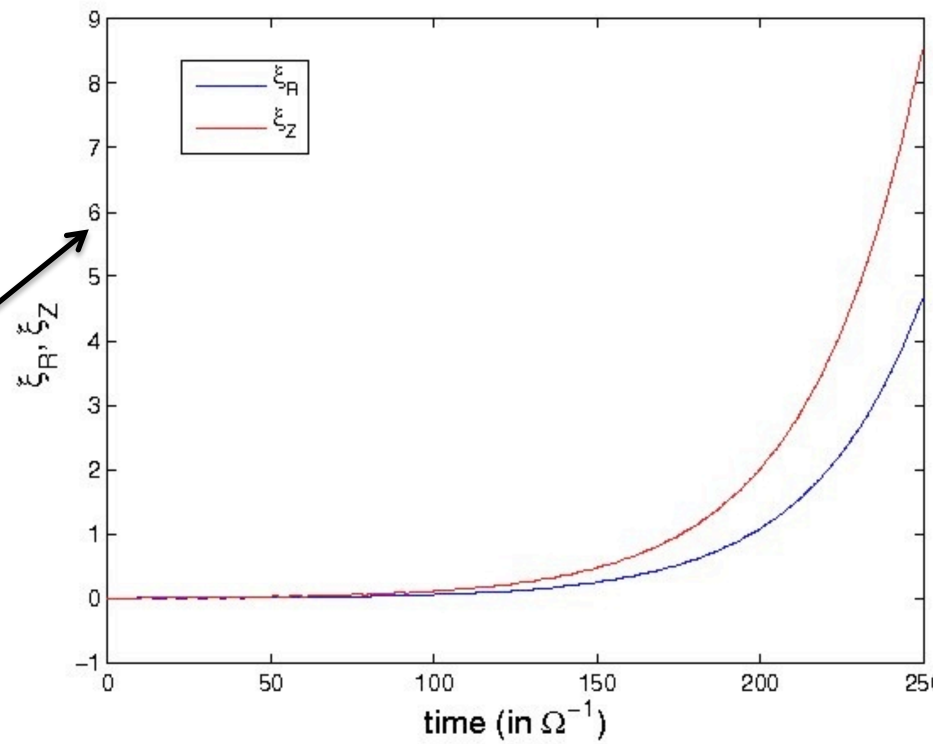


Linear growth rate contours. Solid line is Ω_R / Ω_Z .
(Latter 2009).

Nonaxisymmetric evolution. (Latter 2009)



Without kv_A.



With kv_A.

SUMMARY & SYNTHESIS

1. A dominant balance of the vorticity equation corresponding to a thermal wind balance seems to hold in much of the SCZ.

2. $\partial S/\partial\theta > \partial S/\partial \ln r$, just as seen in Ω contours.

3. TWE equation may be solved exactly with $S=S(\Omega)$. Produces isorotation contours in broad agreement with helioseismology.

4. As it happens, $S=S(\Omega)$ corresponds precisely to marginal stability of axisymmetric, baroclinic, magnetized modes in rotating gas. Coincidence?

SUMMARY & SYNTHESIS

5. As it happens, nonaxisymmetric modes evolve toward $k_R / k_z = \partial_R\Omega / \partial_Z\Omega$, which neutralizes both $D\Omega$ and DS when $S=S(\Omega)$. Coincidence?

6. The gross dynamical ("Batman isotachs") and thermal (adiabatic) features of the SCZ are a consequence of marginalizing the dominant magnetobaroclinic linear unstable modes of the system.

SUMMARY & SYNTHESIS

7. Need to resolve $(kv_A)^2 = \partial\Omega^2 / \partial \ln R$ wavelengths , nominally difficult, not impossible. Can surely fudge parameters to bring into computational domain. Calibration with linear dispersion relation is essential.

8. Ideas are generic, simple. For the future, hope is that they will prove to be useful for problems they were not designed to solve directly, e.g. latitude dependence of dynamo cycle $\Leftrightarrow N^2(r, \theta)$.