# Horizon Surface Gravity in Black Hole Binaries

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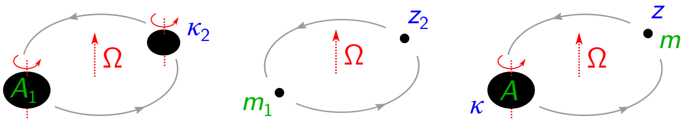


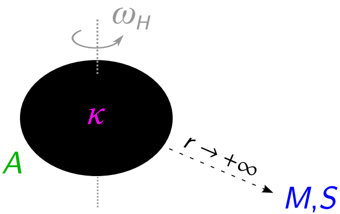
# Black hole uniqueness theorem in GR

[Israel 1967; Carter 1971; Hawking 1973; Robinson 1975]

- The only stationary vacuum black hole solution is the Kerr solution of mass $M$ and angular momentum $S$

*"Black holes have no hair."* (J. A. Wheeler)

- Black hole event horizon $\mathcal{H}$ characterized by:
  - Angular velocity $\omega_H$
  - Surface gravity $\kappa$
  - Surface area $A$

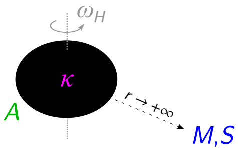


# The laws of black hole mechanics

[Hawking 1972; Bardeen, Carter & Hawking 1973]

- Zeroth law of mechanics:

$$\kappa = \text{const.} \quad (\text{on } \mathcal{H})$$

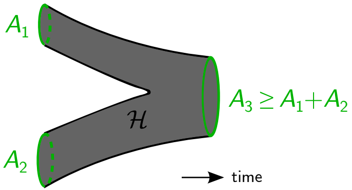



- First law of mechanics:

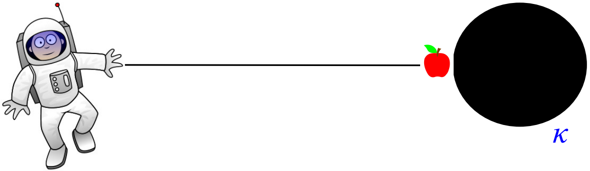
$$\delta M = \omega_H \, \delta S + \frac{\kappa}{8\pi} \, \delta A$$

- Second law of mechanics:

$$\delta A \geq 0$$



# What is the horizon surface gravity?



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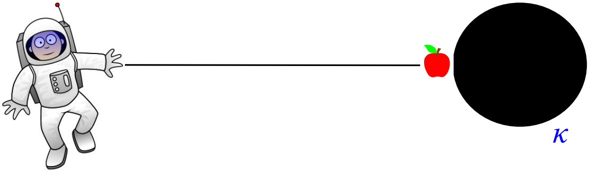


- For an event horizon $\mathcal{H}$ generated by a Killing field $k^\alpha$,

$$\kappa^2 \equiv \frac{1}{2} \left(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha\right)\Big|_{\mathcal{H}}$$

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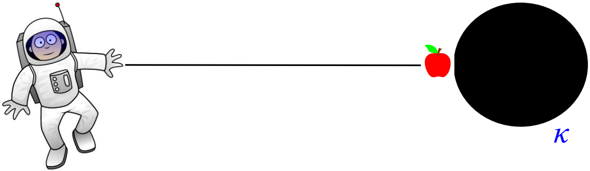


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- For a Schwarzschild black hole of mass $M$, this yields

$$\kappa = \frac{1}{4M} = \frac{GM}{R_S^2}$$

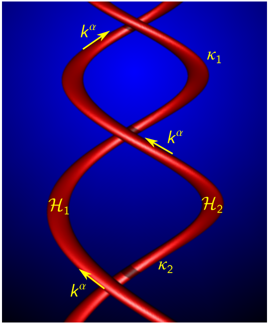
# Zeroth law of *binary* mechanics

[Friedman, Uryū & Shibata 2002]

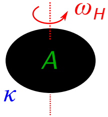
- Black hole spacetimes with *helical* Killing vector field $k^\alpha$
- On each component $\mathcal{H}_a$ of the horizon, the expansion and shear of the geodesic generators vanish
- Generalized rigidity theorem: $\mathcal{H} = \bigcup_a \mathcal{H}_a$ is a Killing horizon
- *Constant* horizon surface gravity

$$\kappa_a^2 = \frac{1}{2}(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha)\big|_{\mathcal{H}_a}$$

- The binary black hole system is in a state of *corotation*

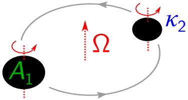


# First laws of *binary* mechanics



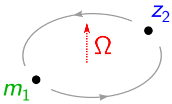

$$\delta M = \omega_H \, \delta S + \frac{\kappa}{8\pi} \, \delta A$$

[Bardeen et al. 1973]




$$\delta M = \Omega \, \delta J + \sum_a \frac{\kappa_a}{8\pi} \, \delta A_a$$

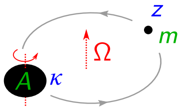
[Friedman et al. 2002]




$$\delta M = \Omega \, \delta J + \sum_a z_a \, \delta m_a$$

[Le Tiec et al. 2012]

[Blanchet et al. 2013]

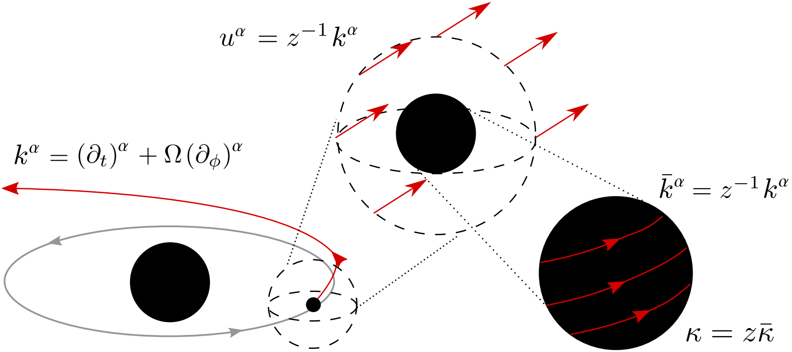



$$\delta M = \Omega \, \delta J + \frac{\kappa}{8\pi} \, \delta A + z \, \delta m$$

[Gralla & Le Tiec 2013]

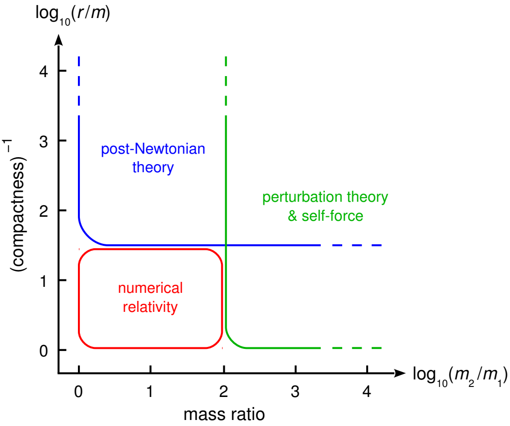
# Surface gravity and redshift

[Pound 2015 (unpublished)]




(Credit: Zimmerman, Lewis & Pfeiffer 2016)

# Source modelling for compact binaries



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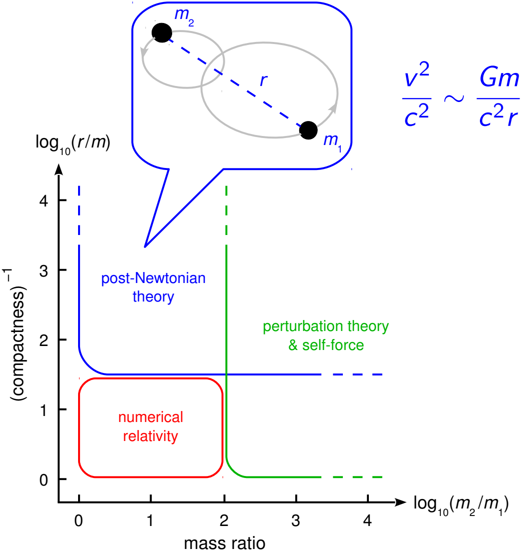
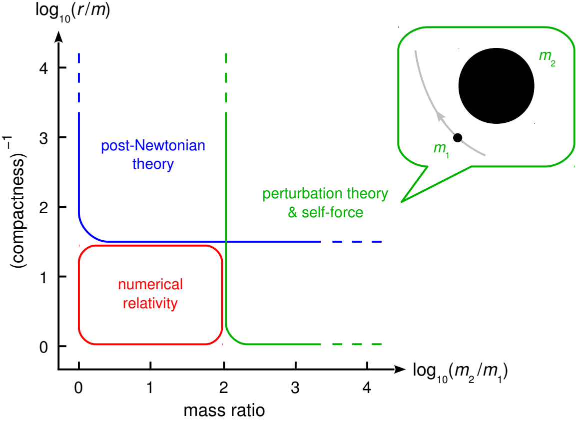



$$\frac{v^2}{c^2} \sim \frac{Gm}{c^2 r} \ll 1$$

# Source modelling for compact binaries

$$q \equiv \frac{m_1}{m_2} \ll 1$$



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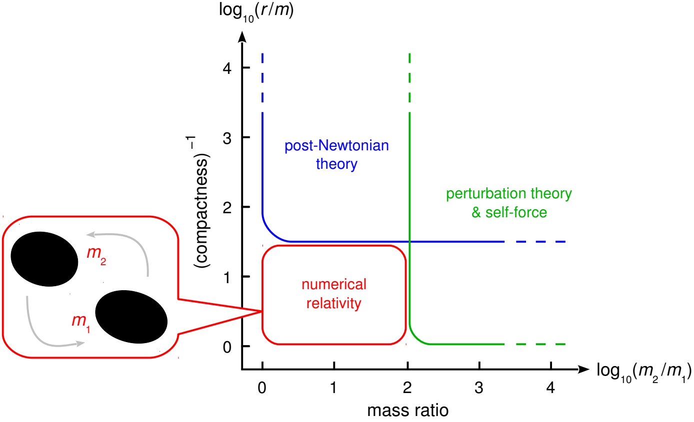

# Quasi-equilibrium initial data

- 3+1 decomposition of the metric:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

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$$\mathcal{L}_k\, g_{\alpha\beta} = 0 \quad \text{with} \quad k^\alpha = (\partial_t)^\alpha + \Omega\,(\partial_\phi)^\alpha$$

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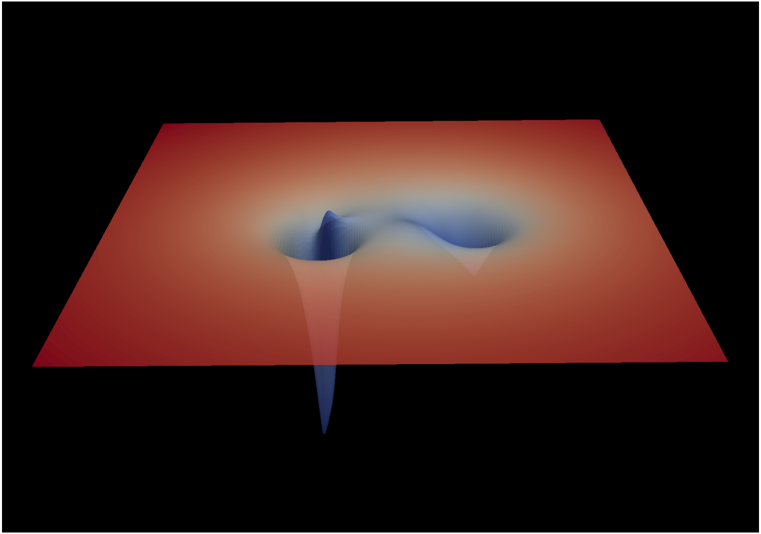
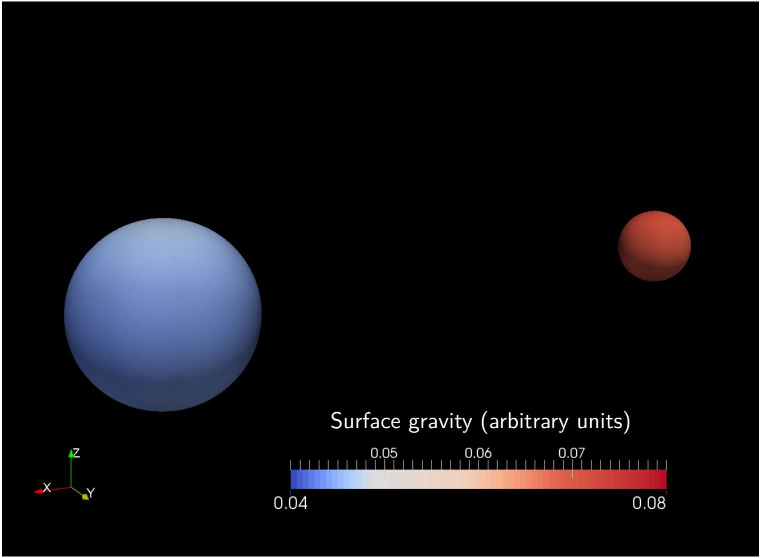
- Impose vanishing linear momentum to find rotation axis

# Curvature and lapse for mass ratio 2 : 1



# Surface gravity for mass ratio 2 : 1



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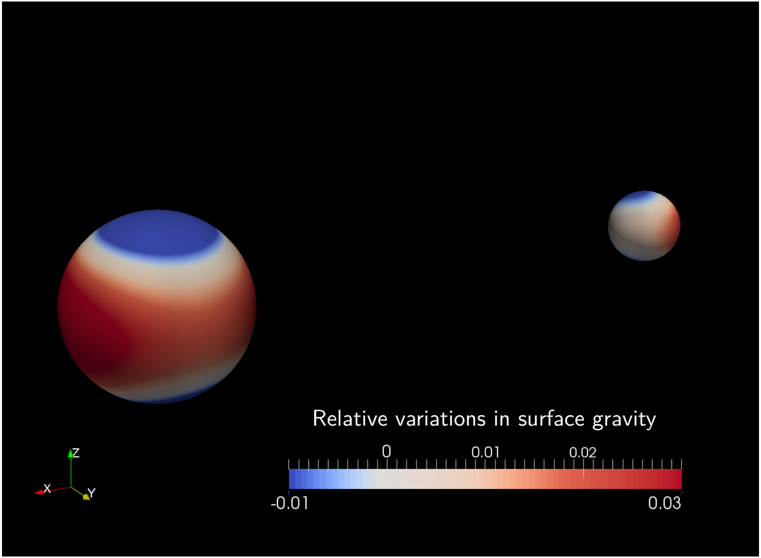
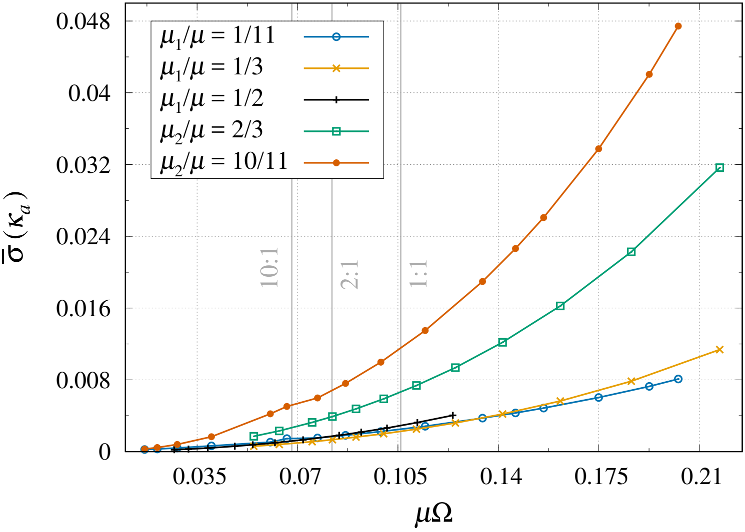
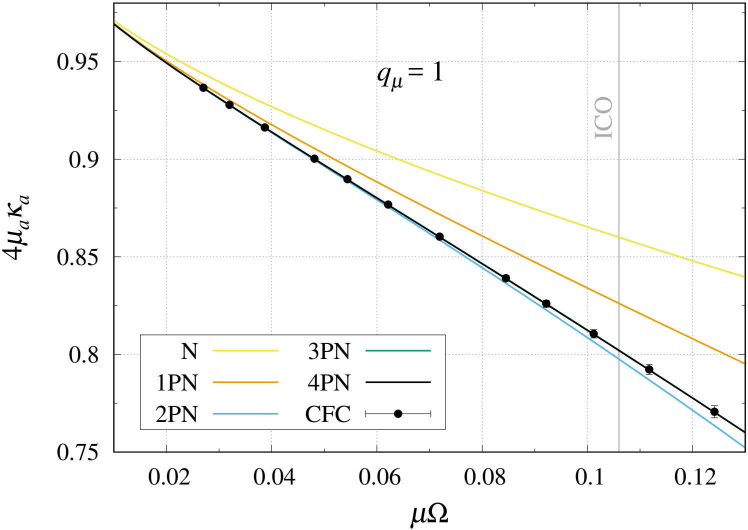


# Variations in horizon surface gravity



# Surface gravity *vs* orbital frequency



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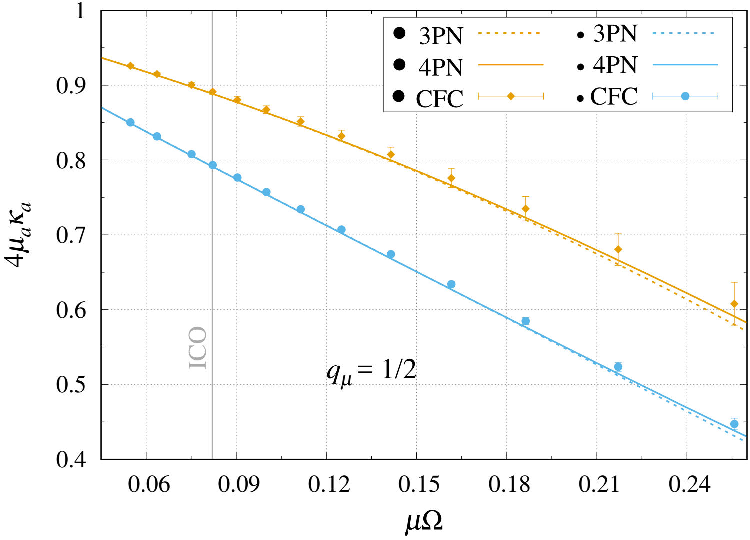

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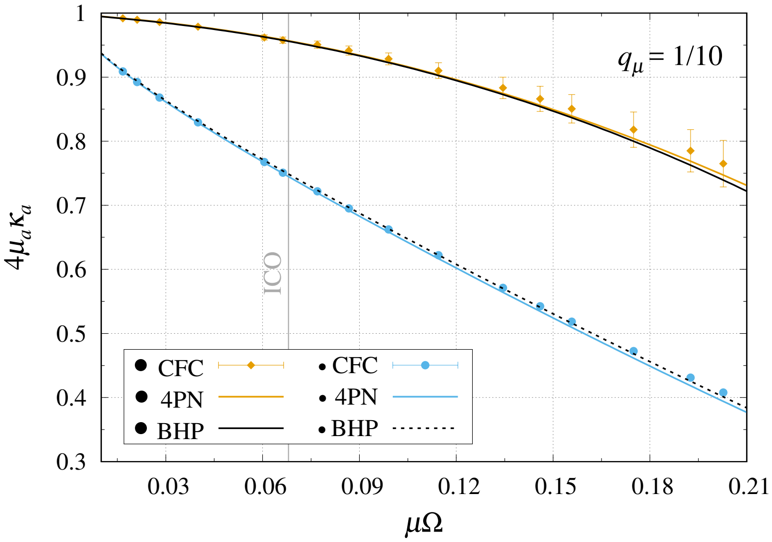

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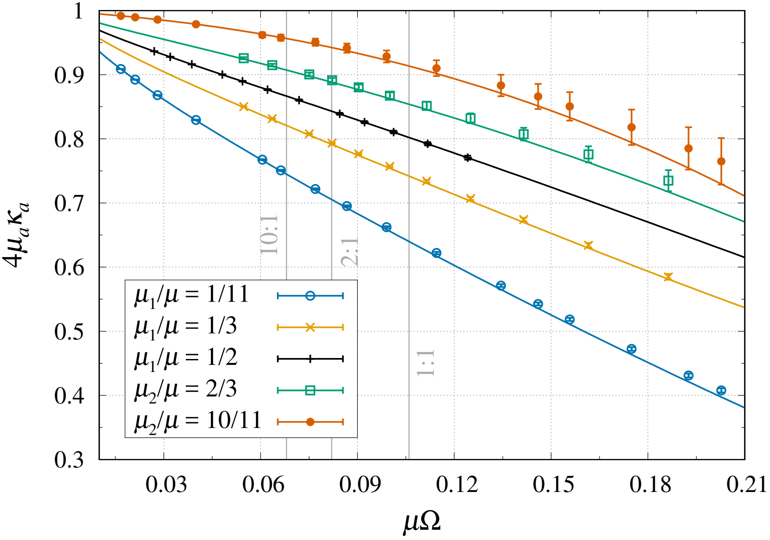
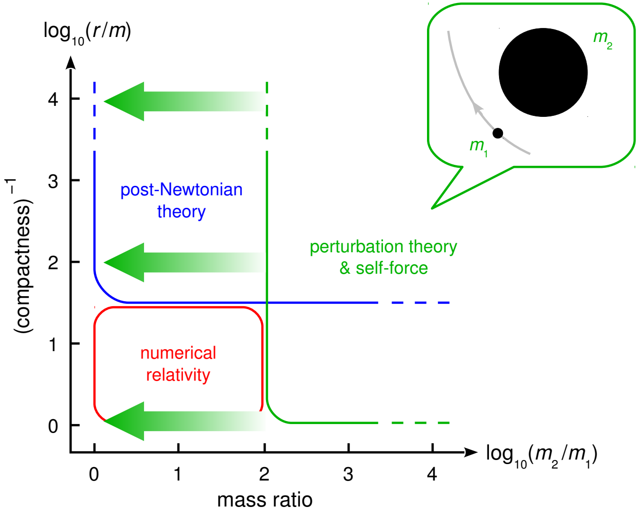


# Perturbation theory for comparable masses



# Summary

- The celebrated laws of black hole (BH) mechanics have been extended to binary BH systems
- In corotating binaries, the surface gravity $\kappa_a$ is constant
- We computed $\kappa_a(\Omega)$ from quasi-equilibrium initial data for corotating BH binaries with comparable masses
- We compared those numerical results to the analytical predictions from the PN approximation and linear BH perturbation theory and found excellent agreement

# Prospects

- Perturbation theory may prove useful to build templates for IMRIs and even comparable-mass binaries

# Additional Material

# Why does BHPT perform so well?

- In perturbation theory, one traditionally expands as

$$f(\Omega; m_a) = \sum_{k=0}^{k_{\text{max}}} a_k(m_2\Omega)\, q^k \quad \text{where} \quad q \equiv m_1/m_2 \in [0,1]$$

- However, most physically interesting relationships $f(\Omega; m_a)$ are symmetric under exchange $m_1 \longleftrightarrow m_2$
- Hence, a better-motivated expansion is

$$f(\Omega; m_a) = \sum_{k=0}^{k_{\text{max}}} b_k(m\Omega)\, \nu^k \quad \text{where} \quad \nu \equiv m_1 m_2/m^2 \in [0,1/4]$$

- In a PN expansion, we have $b_n = \mathcal{O}\big(1/c^{2n}\big) = n\text{PN} + \cdots$

## Why does BHPT perform so well?

- In perturbation theory, each surface gravity is expanded as

$$4\mu_1\kappa_1 = a(\mu_2\Omega) + q\, b(\mu_2\Omega) + \mathcal{O}(q^2)$$
$$4\mu_2\kappa_2 = c(\mu_2\Omega) + q\, d(\mu_2\Omega) + \mathcal{O}(q^2)$$

- From the first law we know that the general form is

$$4\mu_a\kappa_a = \sum_{k\geqslant 0} \nu^k f_k(\mu\Omega) \pm \sqrt{1-4\nu} \sum_{k\geqslant 0} \nu^k g_k(\mu\Omega)$$

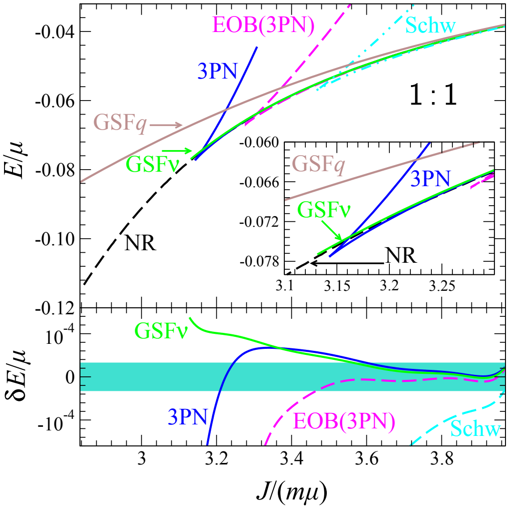
- Each surface gravity can thus be rewritten as

$$\begin{aligned} 4\mu_a\kappa_a = A(\mu\Omega) &\pm B(\mu\Omega)\sqrt{1-4\nu} + C(\mu\Omega)\,\nu \\ &\pm D(\mu\Omega)\,\nu\sqrt{1-4\nu} + \mathcal{O}(\nu^2) \end{aligned}$$

- Expand to linear order in $q$ and match $\to$ $A$, $B$, $C$, $D$

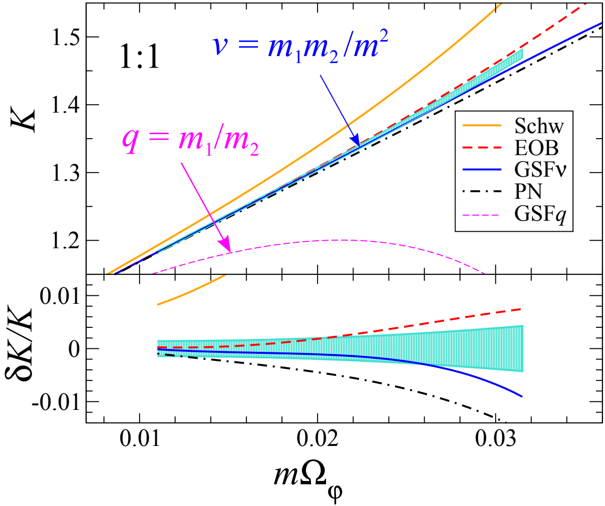
# Binding energy *vs* angular momentum

[Le Tiec, Barausse & Buonanno 2012]



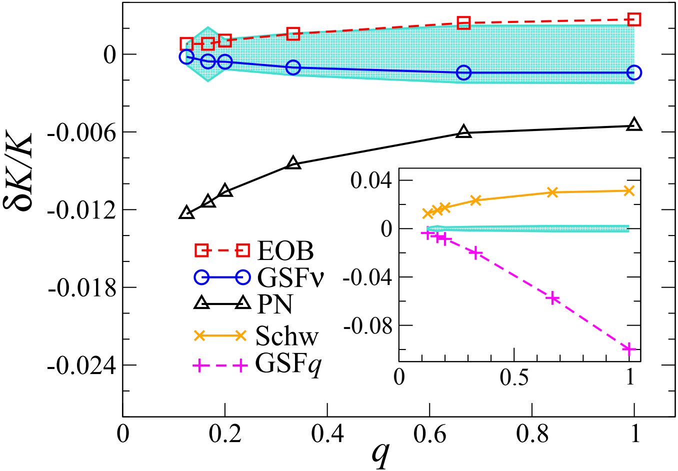
# Periastron advance *vs* orbital frequency

[Le Tiec, Mroué *et al.* 2011]



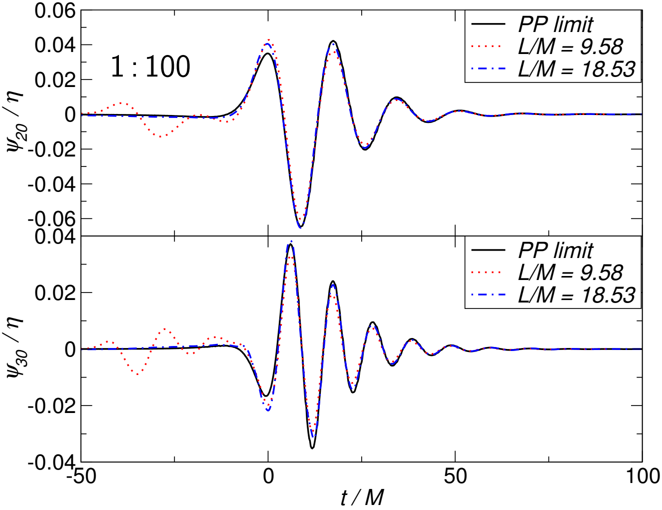
# Periastron advance *vs* mass ratio

[Le Tiec, Mroué *et al.* 2011]



# Waveform from head-on collision

[Sperhake, Cardoso *et al.* 2011]



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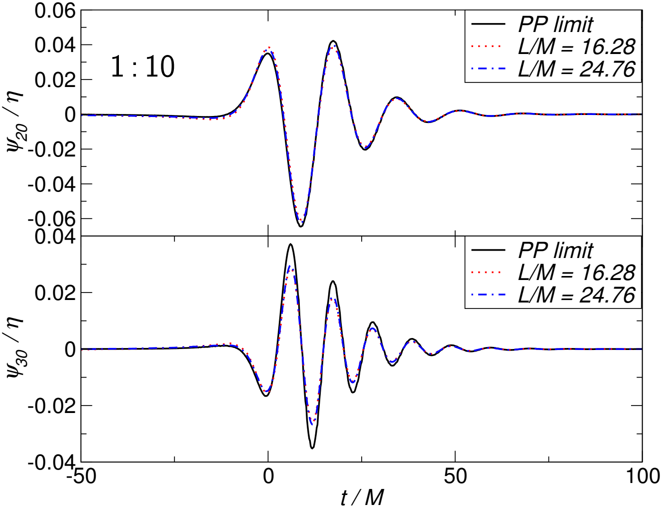
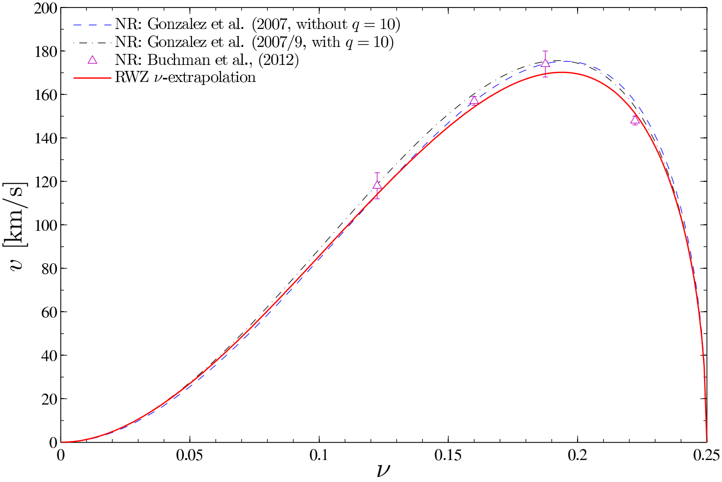


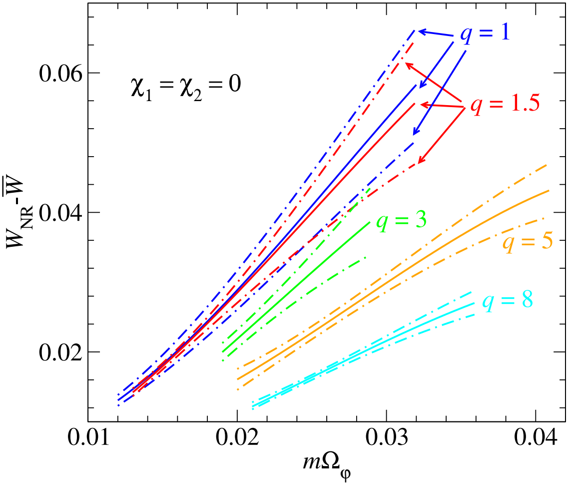
# Recoil velocity *vs* symmetric mass ratio

[Nagar 2013]



# Self-force from numerical relativity

[Le Tiec, Buonanno *et al.* 2013]



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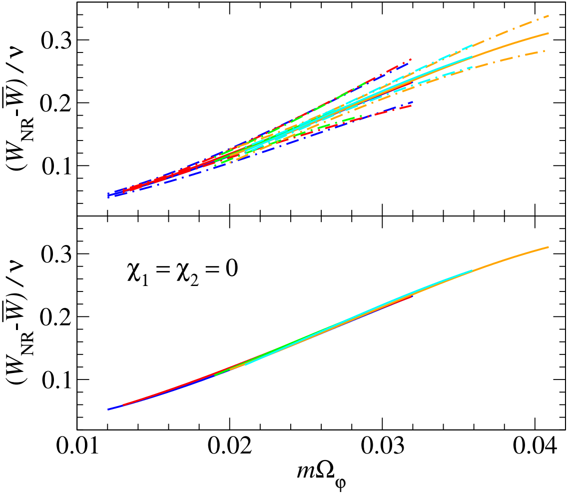

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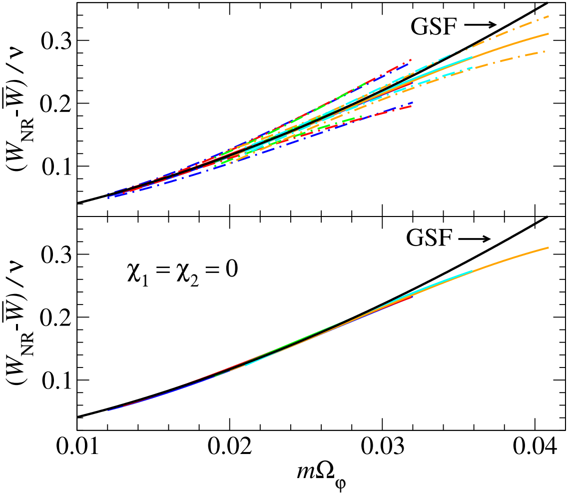

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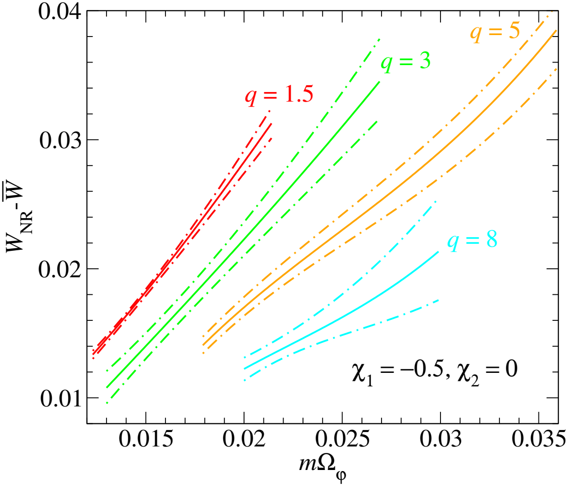

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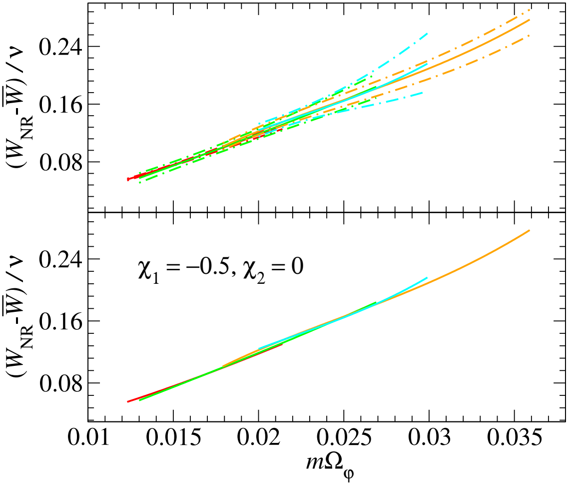

# Prediction confirmed!

[van de Meent 2017]