Entropy theorems in classical mechanics, general relativity, and the gravitational two-body problem

Phys. Rev. D **94**, 064049 (2016)

Marius Oltean[1,2,3,4,5,*], Luca Bonetti[3,4,5], Alessandro D.A.M. Spallicci[3,4,5] and Carlos F. Sopuerta[1]

[1] *Institut de Ciències de l'Espai (IEEC-CSIC), Campus Universitat Autònoma de Barcelona, Spain*
[2] *Departament de Física, Facultat de Ciències, Universitat Autònoma de Barcelona, Spain*
[3] *Observatoire des Sciences de l'Univers en région Centre, Université d'Orléans, France*
[4] *Pôle de Physique, Collegium Sciences et Techniques, Université d'Orléans, France*
[5] *Laboratoire de Physique et Chimie de l'Environnement et de l'Espace, CNRS, France*

* olteanice.cat

ABSTRACT

In classical Hamiltonian theories, entropy may be understood either as a statistical property of canonical systems or as a mechanical property, that is, as a monotonic function of the phase space along trajectories. In classical mechanics, there are theorems which have been proposed for proving the nonexistence of entropy in the latter sense. We explicate, clarify, and extend the proofs of these theorems to some standard matter (scalar and electromagnetic) field theories in curved spacetime, and then we show why these proofs fail in general relativity; due to properties of the gravitational Hamiltonian and phase space measures, the second law of thermodynamics holds. As a concrete application, we focus on the consequences of these results for the gravitational two-body problem, and in particular, we prove the noncompactness of the phase space of perturbed Schwarzschild-Droste spacetimes. We thus identify the lack of recurring orbits in phase space as a distinct sign of dissipation and hence entropy production.

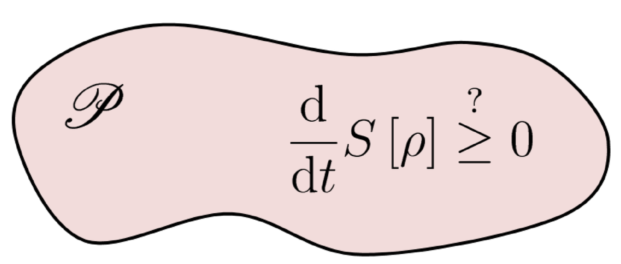
INTRODUCTION

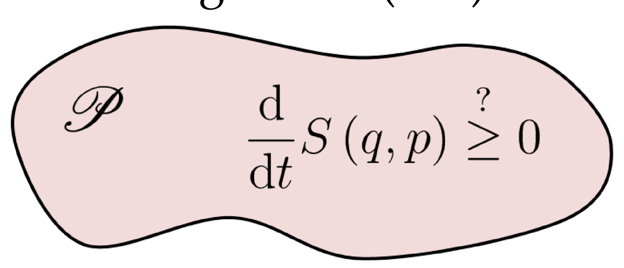
In the two-body problem in general relativity (GR), we expect entropy increase from gravitational wave emission. Yet, there is presently little consensus on the meaning and computational prescription of "the entropy of a gravitational system" in general, and still less on the problem of why it should obey the second law of thermodynamics.

We here revisit and clarify the approaches to this problem in classical mechanics (CM), and address their (non-)applicability to GR [1].

THE PROBLEM OF THE SECOND LAW

We work with Hamiltonian theories on a phase space $\mathscr{P}$ (or a reduced phase space when there exist constraints in the theory, *e.g.* in electromagnetism (EM) or GR).





	"Statistical" problem: *Does there exist a* ***functional*** $S[\rho]$ ***of a probability density*** ρ ***on*** $\mathscr{P}$ *which monotonically increases in time?*	*"Mechanical" problem:* *Does there exist a* ***phase space function*** $S:\mathscr{P}\to\mathbb{R}$ *which monotonically increases along trajectories?*
CM	Yes. [2]	No.
GR	Maybe?...	Yes.

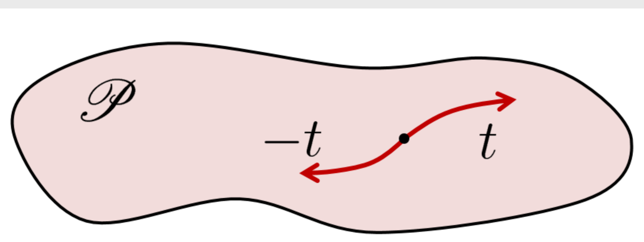
We are interested in the ***"mechanical" problem*** in CM/GR, *i.e.* proving/answering:

- **Theorem:** *In CM,* $\nexists\ S:\mathscr{P}\to\mathbb{R}$ *monotonically increasing along trajectories.*
- **Question:** *In GR, why do we expect the conditions of this theorem to not hold?*

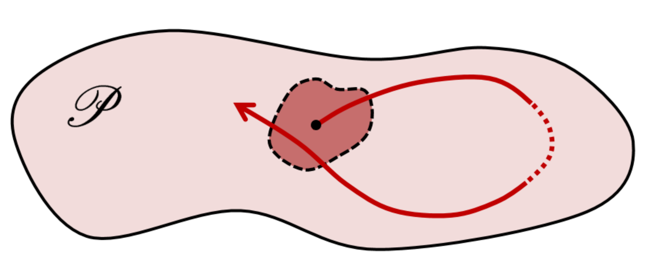
APPROACHES TO THE "MECHANICAL" PROBLEM IN CM

Some well-known approaches to this problem:

- *Loschmidt reversibility argument* [3]: the canonical equations of motion are time-reversal symmetric.



- *Poincaré recurrence theorem* [4]: any canonical system in a bounded phase space will return arbitrarily close to its initial state (an unbounded number of times).



We consider here some less well-known approaches:

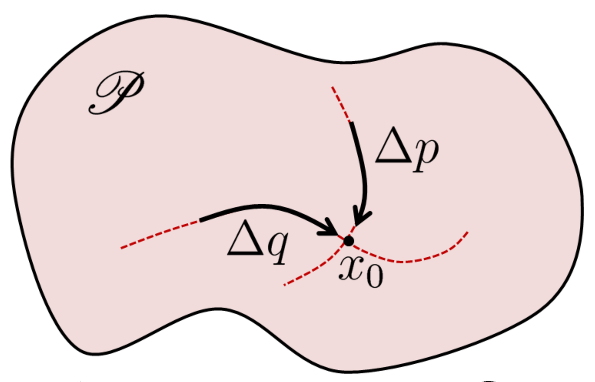
1. ***Perturbative approach***: sketched by Poincaré [5]; we carried out the full proof [1].
2. ***Topological approach***: proof by Olsen [6].

PERTURBATIVE APPROACH

CM *Idea of the proof* [1]: We Taylor expand the Poisson bracket,

$$\dot{S}=\{S,H\}=\sum_{k=1}^{N}\left(\frac{\partial H}{\partial p_k}\frac{\partial S}{\partial q_k}-\frac{\partial H}{\partial q_k}\frac{\partial S}{\partial p_k}\right)$$

about a hypothetical "equilibrium" point x_0 and we get a **contradiction** with $\dot{S}>0$ away from it. (We require some assumptions on H, but **no** topological assumptions on $\mathscr{P}$).

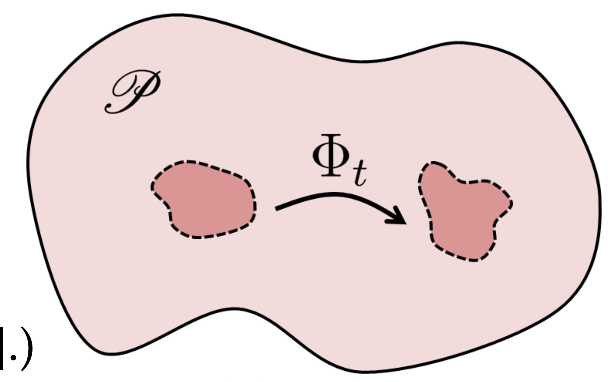


GR *Following the same approach* [1]:

- We proved the theorem also for a scalar field and EM in curved spacetime.
- The theorem **does not** work in GR because of the curvature properties (in phase space) of the gravitational (vacuum) Hamiltonian.

TOPOLOGICAL APPROACH

CM *Proof by Olsen* [6]: If $\mathscr{P}$ is compact and invariant, the volume integral of S in $\mathscr{P}$ is invariant.



GR *Following the same approach* [1]:

- The proof **does not** work because, in general, the phase space of GR is non-compact. (See, *e.g.*, [7].)
- We explicitly proved, in particular, the non-compactness of the phase space of perturbed Schwarzschild(-Droste) spacetimes.

CONCLUSIONS

Summary of results:

- We have provided a full rigorous proof in CM, via the *perturbative* approach, for the non-existence of ("mechanical") entropy production; we have also extended this to some matter (scalar and EM) field theories in curved spacetime.
- Both the *perturbative* and *topological* proofs are not applicable in GR, and specifically, the latter for perturbed Schwarzschild(-Droste) spacetimes.

Future work:

- Necessary **and sufficient** conditions for the perturbative approach proof?
- Proof that a general definition of entropy in GR obeys the second law?

[1] MO, LB, ADAMS and CFS, Phys. Rev. D **94**, 064049 (2016).
[2] C. Villani, in *Boltzmann's Legacy (ESI Lectures in Mathematics and Physics)* (Eur. Math. Soc., 2008), p. 129.
[3] J. J. Loschmidt, Sitzungsber. Akad. Wiss. Wien Math. Naturwiss. Kl. Abt. 1 **73**, 128 (1876).
[4] H. Poincaré, Acta Math. **13**, 1 (1890).
[5] H. Poincaré, C. R. Seances Acad. Sci. (Paris) **108**, 550 (1889).
[6] E. T. Olsen, Found. Phys. Lett. **6**, 327 (1993).
[7] J. S. Schiffrin and R. M. Wald, Phys. Rev. D **86**, 023521 (2012).