Power spectrum estimation: lessons from CMB polarization

Anthony Challinor

Institute of Astronomy
and
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
a.d.challinor@ast.cam.ac.uk

Thanks to: Micheal Brown, George Efstathiou, Lindsay King, Antony Lewis, Dipak Munshi, Patrick Valageas

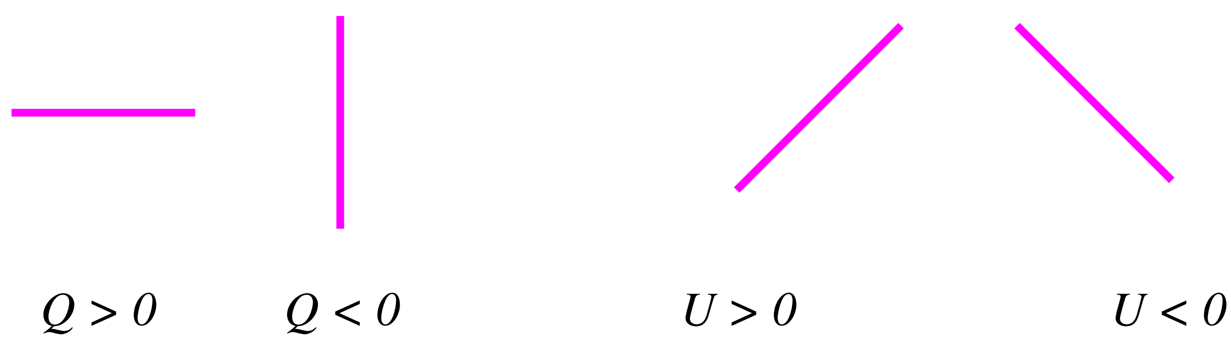
From giant arcs to CMB lensing; IAP 6 July 2007

LINEAR POLARIZATION

- Linear polarization described by transverse, STF tensor

$$\mathcal{P}_{ab} \equiv \begin{pmatrix} \frac{1}{2}\langle E_x^2 - E_y^2 \rangle & \langle E_x E_y \rangle \\ \langle E_x E_y \rangle & -\frac{1}{2}\langle E_x^2 - E_y^2 \rangle \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}$$

  - C.f. shear tensor: $Q \leftrightarrow \gamma_1$, $U \leftrightarrow \gamma_2$



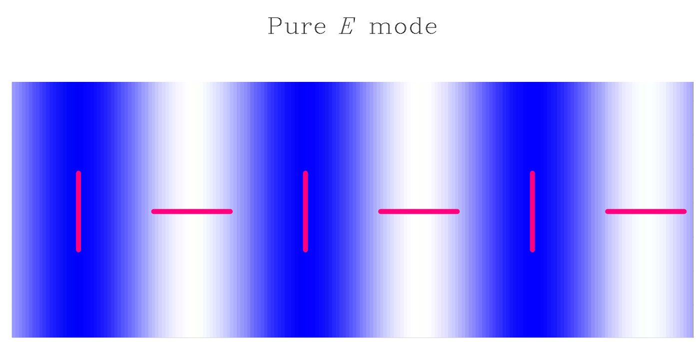
- Under right-handed rotation of x and y through ψ about propagation direction (z)

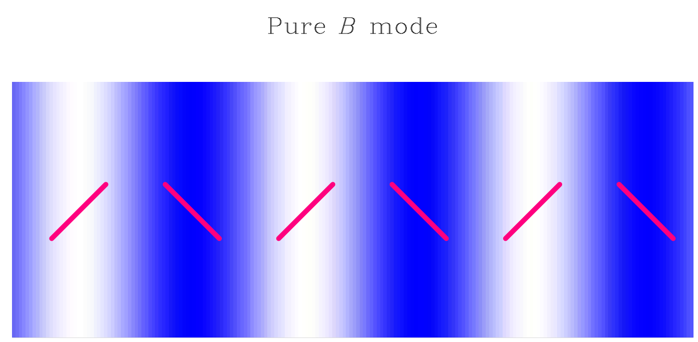
$$Q \pm iU \rightarrow (Q \pm iU)e^{\mp 2i\psi} \quad \Rightarrow \quad Q + iU \text{ is spin -2}$$

E- AND B-MODES OF POLARIZATION

- Unique decomposition of $\mathcal{P}_{ab}(\hat{\boldsymbol{n}})$ on sphere into P_E electric (gradient) part and P_B magnetic (curl) part:

$$\mathcal{P}_{ab}(\hat{\boldsymbol{n}}) = \nabla_{\langle a}\nabla_{b\rangle}P_E + \epsilon^c{}_{(a}\nabla_{b)}\nabla_c P_B$$

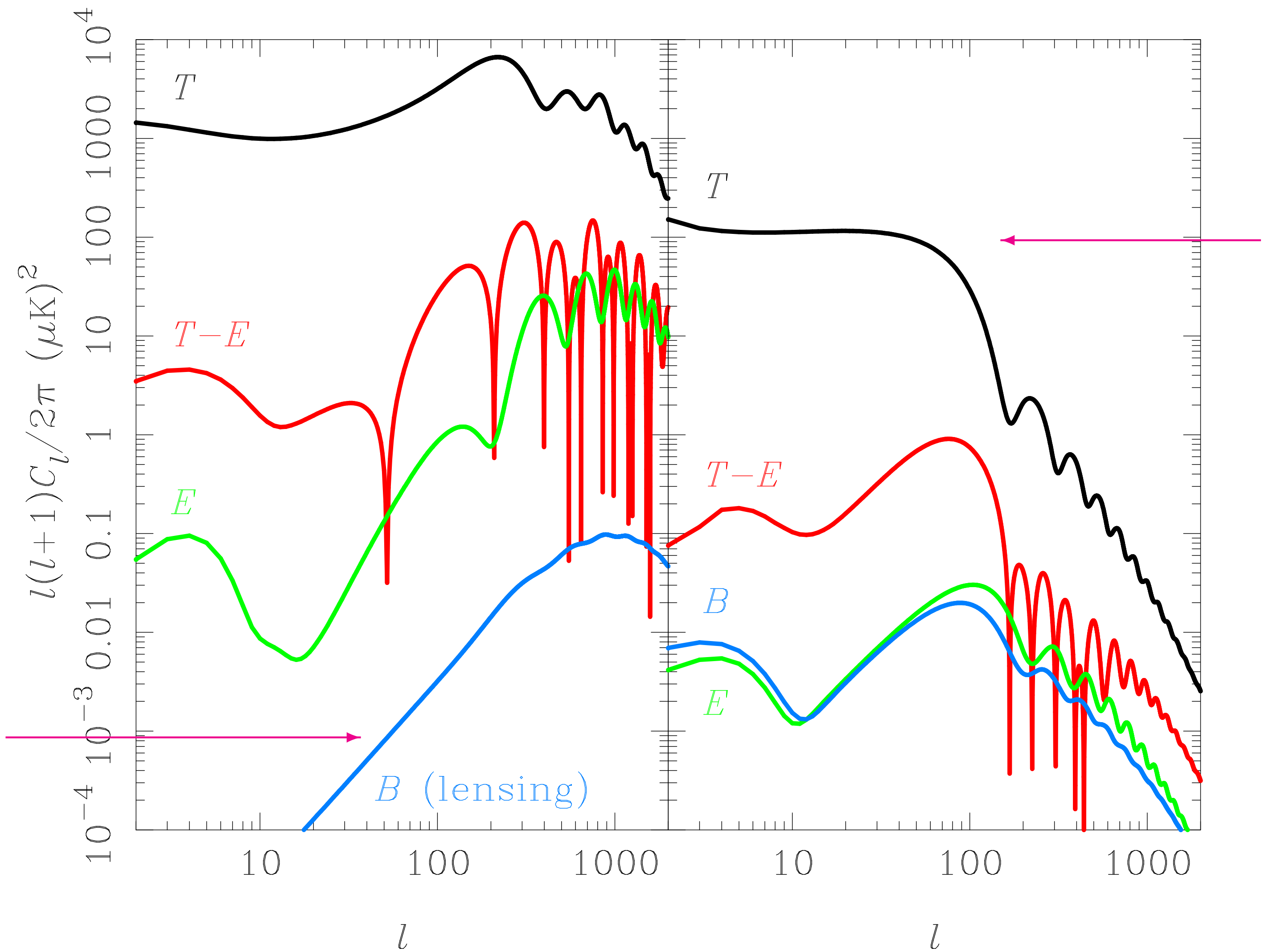




- Linear scalar fluctuations produce only E-mode polarization (Kamionkowski et al. 1997; Seljak & Zaldarriaga 1997)
- Gravity waves produce roughly equal E and B
  - C.f. cosmic shear: B-modes are from systematics (and second-order effects)

PHYSICS OF CMB POLARIZATION

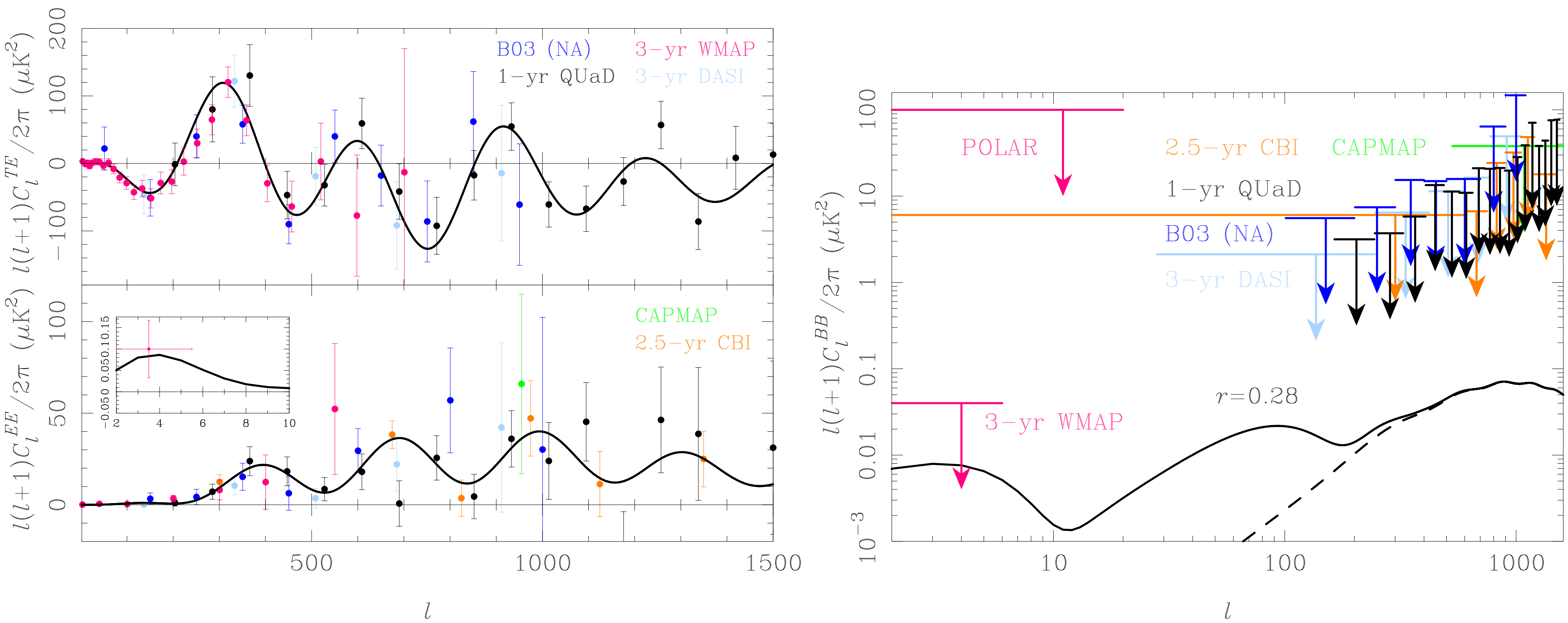
Lens-induced B modes ($\sqrt{C_l^B} \approx 1.3\,$nK)



Effects only on large scales since gravity waves damp inside horizon

- Pol. $\sim 5\,\mu$K generated at recombination and (on large scales) at reionization

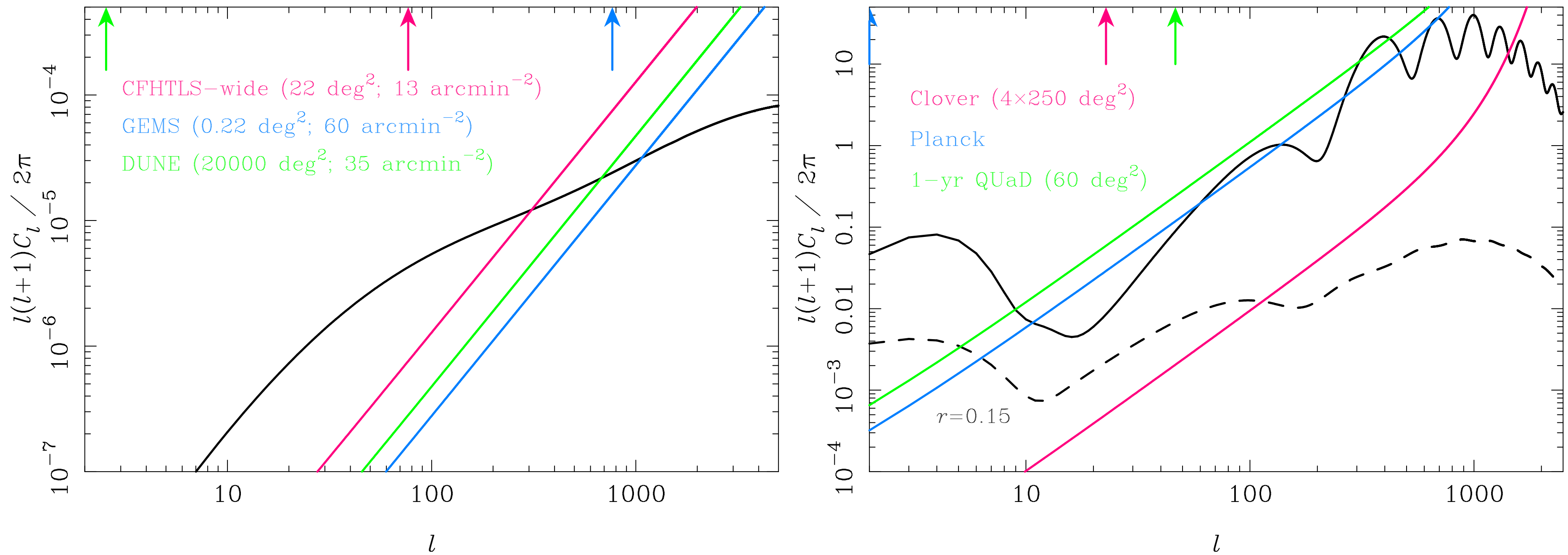
CURRENT RESULTS



- Large-angle E-modes $\Rightarrow \tau = 0.09 \pm 0.03$ (Page et al. 2006)
- 95% limit $r < 0.28$ from CMB and LSS (Spergel et al. 2006) $\Rightarrow$ r.m.s. of primordial B-modes $< 200\,\mathrm{nK}$

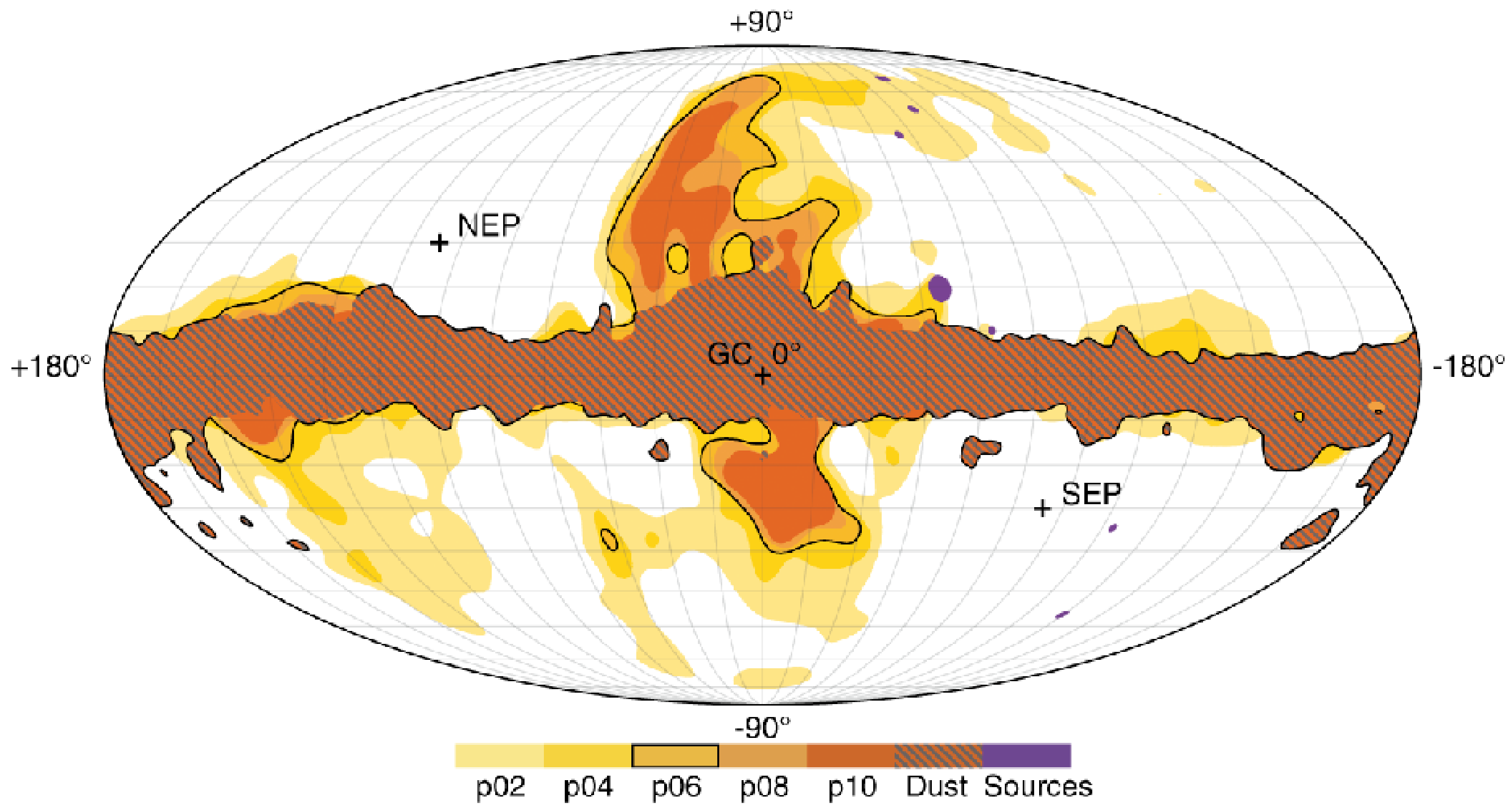
SPECTRAL COMPARISONS

- Both lensing and CMB E-mode spectra very blue
- CMB not yet sample-variance limited; c.f. cosmic shear
- Non-linearities important for cosmic shear below $\sim 10\,\mathrm{arcmin}$ before shot noise dominates (c.f. lens-induced B-modes)



SURVEY GEOMETRY

- Mask Galactic foregrounds and bright extragalactic sources



MAXIMUM-LIKELIHOOD POWER SPECTRUM ESTIMATION: I*

- Pixelised noisy data: $\boldsymbol{x} \equiv \{(Q+iU)(\hat{\boldsymbol{n}}_1), (Q-iU)(\hat{\boldsymbol{n}}_1), \ldots, (Q-iU)(\hat{\boldsymbol{n}}_{N_{\text{pix}}})\}$ with covariance $\boldsymbol{C} = \boldsymbol{S} + \boldsymbol{N}$ where signal correlation $\boldsymbol{S}$

$$\langle (Q+iU)(\hat{\boldsymbol{n}}_1)(Q+iU)(\hat{\boldsymbol{n}}_2)\rangle = \frac{1}{2}e^{-2i(\alpha+\gamma)}\sum_l \frac{2l+1}{4\pi}b_l p_l (C_l^E - C_l^B) d^l_{2\,-2}(\beta)$$

$$\langle (Q-iU)(\hat{\boldsymbol{n}}_1)(Q+iU)(\hat{\boldsymbol{n}}_2)\rangle = \frac{1}{2}e^{2i(\alpha-\gamma)}\sum_l \frac{2l+1}{4\pi}b_l p_l (C_l^E + C_l^B) d^l_{2\,2}(\beta)$$

  – Spherical generalisation of usual flat-sky results in cosmic shear

- Locate maximum of Gaussian $\mathcal{L}$ by N-R or one-step QML (Tegmark 1997):

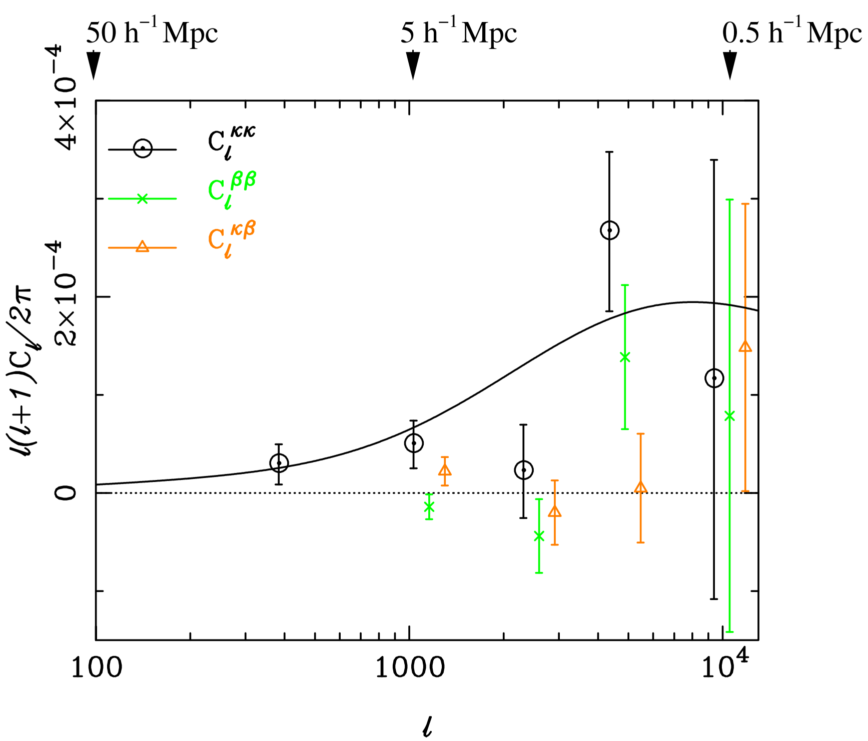
$$\hat{C}_l^R = \frac{1}{2}F^{-1}_{(lR)(l'S)}\text{trace}\left[\boldsymbol{C}^{-1}\frac{\partial \boldsymbol{C}}{\partial C_{l'}^S}\boldsymbol{C}^{-1}(\boldsymbol{x}\boldsymbol{x}^\dagger - \boldsymbol{N})\right]$$

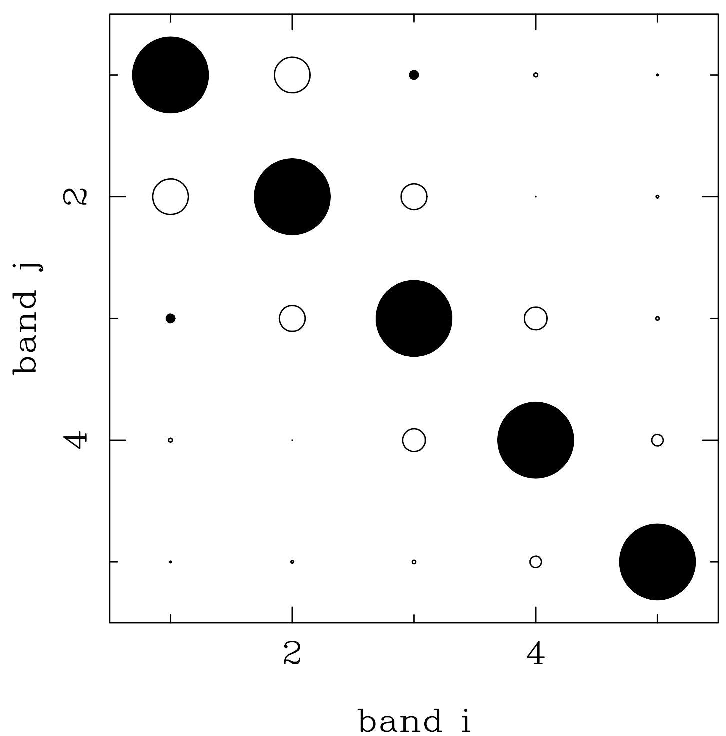
  – Fisher matrix $F_{(lR)(l'S)} \equiv \frac{1}{2}\text{trace}\left[\boldsymbol{C}^{-1}\frac{\partial \boldsymbol{C}}{\partial C_l^R}\boldsymbol{C}^{-1}\frac{\partial \boldsymbol{C}}{\partial C_{l'}^S}\right]$ approximates inverse errors for Gaussian fields

  – QML requires fiducial $\boldsymbol{S}$ but *unbiased* for any choice

*CMB – Bond, Jaffe & Knox (1998); cosmic shear – Hu & White (2001)

- Advantages: optimal on all scales; accounts properly for survey geometry and variation of sampling densities; isolation of E and B modes; errors (at least on large and small scales); "uncorrelated" bandpowers
- Disadvantages: slow [$O(N_{\rm pix}^3)$]
- CMB: interferometers (CBI – Sievers et al. 2006 – and DASI – Leitch et al. 2005); and imaging arrays (MAXIPOL – Wu et al. 2006 – $O(2000)$ pixels)
- Cosmic shear: e.g. COMBO-17 (Brown et al. 2003) and GEMS (Heymans et al. 2005)





PSEUDO-C_l ESTIMATORS*

- Fast [$O(N_{\rm pix}^{3/2})$], robust and widely used
- Unbiased and close to optimal for $l \gg 1/R$ or where noise dominates
- QML: spherical transform and naive spectrum of $\boldsymbol{C}^{-1}\boldsymbol{x}$ → remove noise bias → deconvolve survey geometry
- PC_l replaces optimal $\boldsymbol{C}^{-1}$ weighting with some diagonal $w(\hat{\boldsymbol{n}})$
  - Extract *pseudo-multipoles*, $\tilde{E}_{lm}$ and $\tilde{B}_{lm}$ of weighted data and compress to $\tilde{C}_l$:

$$\tilde{C}_l^E \equiv \frac{1}{2l+1}\sum_m |\tilde{E}_{lm}|^2 \quad , \quad \tilde{C}_l^B \equiv \frac{1}{2l+1}\sum_m |\tilde{B}_{lm}|^2$$

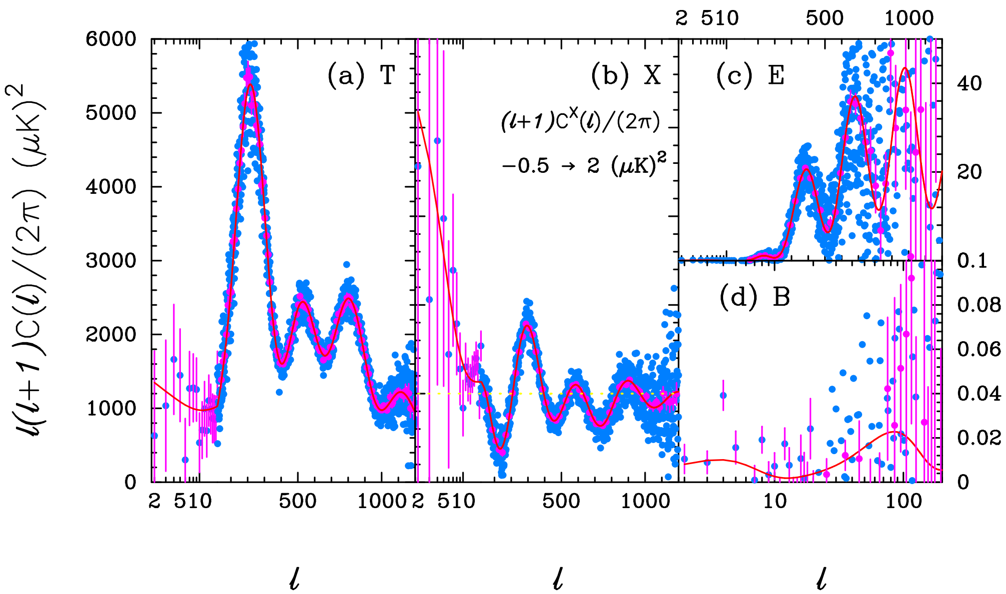
  - Subtract off mean noise level (Monte-Carlo for complicated noise)
  - Deconvolve by inverting analytic coupling matrices (e.g. bandpowers)

*CMB: Wandelt et al. 2001; Hivon et al. 2002

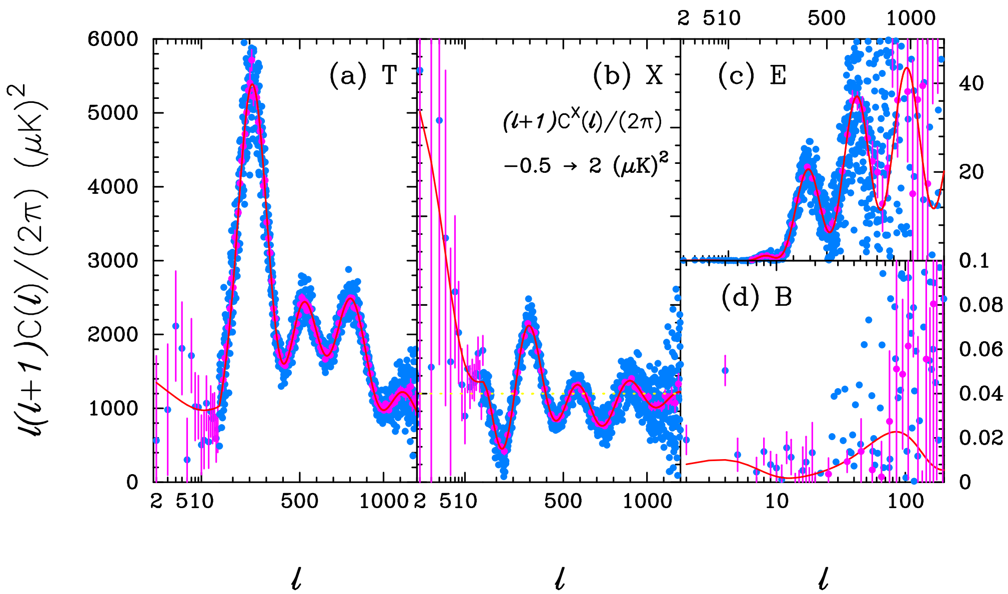
CHOICE OF WEIGHT FUNCTION*

- Uniform where signal dominated
- Inverse variance where noise-dominated
  - But regularise/smooth sharp features (e.g. Planck ecliptic poles)

Equal weight



Inverse-variance



*Efstathiou (2004)

RELATION OF PSEUDO-C_l AND CORRELATION FUNCTIONS*

- Correlation functions with pair-weighting $w(\hat{\boldsymbol{n}}_1)w(\hat{\boldsymbol{n}}_2)$ directly related to $\tilde{C}_l$:

$$\hat{\xi}_{\bar{Q}\bar{Q}}(\beta) = A(\beta)\sum_l (2l+1)\left[\tfrac{1}{2}(d^l_{22} + d^l_{2\,-2})(\beta)\tilde{C}^E_l + \tfrac{1}{2}(d^l_{22} - d^l_{2\,-2})(\beta)\tilde{C}^B_l\right]$$

$$\hat{\xi}_{\bar{U}\bar{U}}(\beta) = A(\beta)\sum_l (2l+1)\left[\tfrac{1}{2}(d^l_{22} - d^l_{2\,-2})(\beta)\tilde{C}^E_l + \tfrac{1}{2}(d^l_{22} + d^l_{2\,-2})(\beta)\tilde{C}^B_l\right]$$

where $A^{-1}(\beta) \equiv \sum_l (2l+1) P_l(\cos\beta) w_l$ involves power spectrum of $w(\hat{\boldsymbol{n}})$

- $\tilde{C}_l$ route faster [$O(N^{3/2}_{\rm pix})$] than direct evaluation [$O(N^2_{\rm pix})$]
- Naturally interpolates over all pair separations in survey

*Chon, AC et al. (2004)

ISOLATING E AND B POWER: I

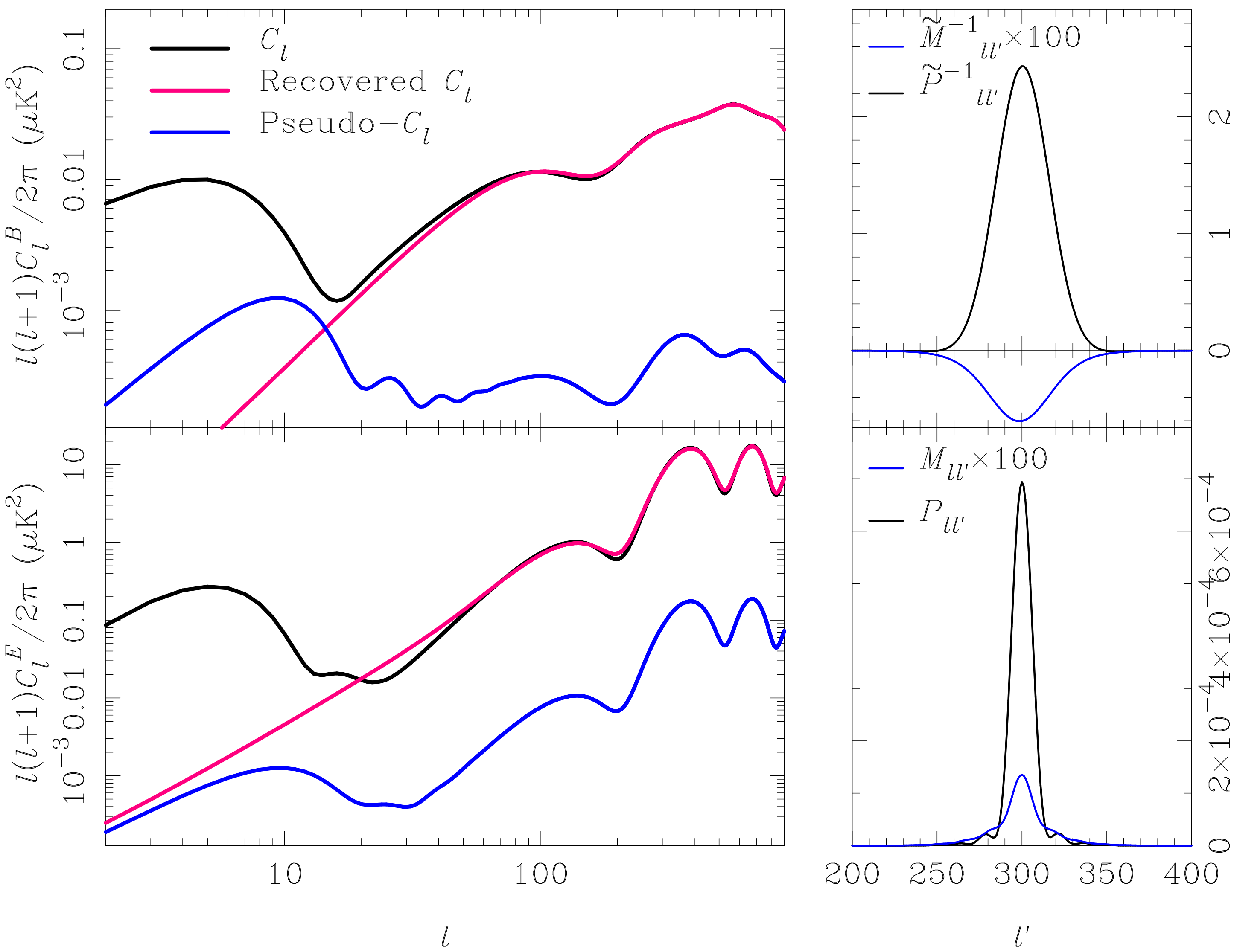
$$\langle \tilde{C}_l^{E/B} \rangle = \sum_{l'} \left(P_{ll'} C_{l'}^{E/B} + M_{ll'} C_{l'}^{B/E} \right)$$

- E- and B-mode power mixed in pC_l
  - For $l \gg 1/R$ mixing of power suppressed by $1/(lR)^2$ (AC & Chon 2005):

$$\sum_{l'} P_{ll'} \approx \frac{1}{4\pi} \int w^2(\hat{\boldsymbol{n}}) d\hat{\boldsymbol{n}} \quad , \quad \sum_{l'} M_{ll'} \approx -\frac{1}{2\pi} \frac{1}{l(l+1)} \int (\nabla w)^2 (\hat{\boldsymbol{n}}) d\hat{\boldsymbol{n}}$$

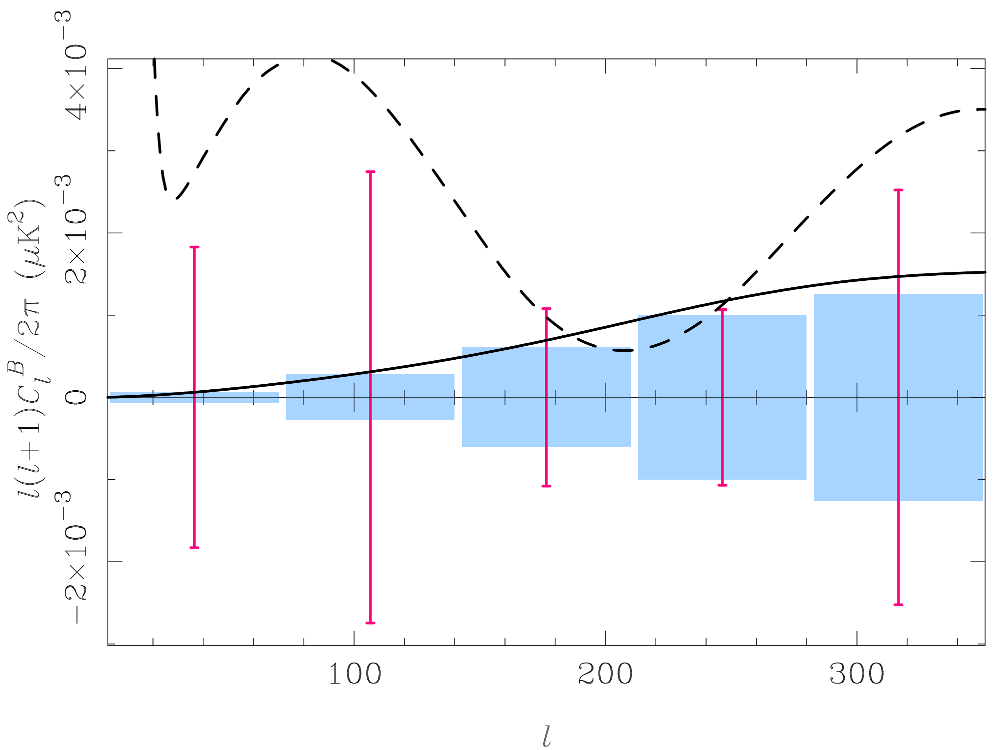
- Invert pC_l to C_l with:
  - Direct inversion of $P \pm M$ if enough sky $\Rightarrow$ unbiased (e.g. WMAP)
  - Inversion with bandpowers (e.g. BOOMERanG, QUaD)
  - Via correlation functions (e.g. SPICE)
- Crittenden et al. (2002) and Schneider et al. (2002) post-processing of $\xi_{\bar{Q}\bar{Q}}$ and $\xi_{\bar{U}\bar{U}}$ to isolate E and B-modes also generalises to spherical geometry (Chon et al. 2004)

ISOLATING E AND B POWER: EXAMPLE



ESTIMATOR VARIANCE AND AMBIGUOUS MODES*

- PC_l estimators *do not* coherently separate E and B-modes
  - Excess *estimator variance* on large scales
  - For surveys of 1–2% of sky (Clover, QUIET etc.) estimator limits minimum-detectable $r \sim 0.05$



Sample variance from C_l^E

Sample variance from C_l^B

*AC & Chon (2005)

ISOLATING B-MODES IN MAPS*

- E-B non-unique over part of sky because kernel of $\nabla^2(\nabla^2+2)$ not empty
- Recover part of *pure-mode* component with B-mode 'aperture mass':

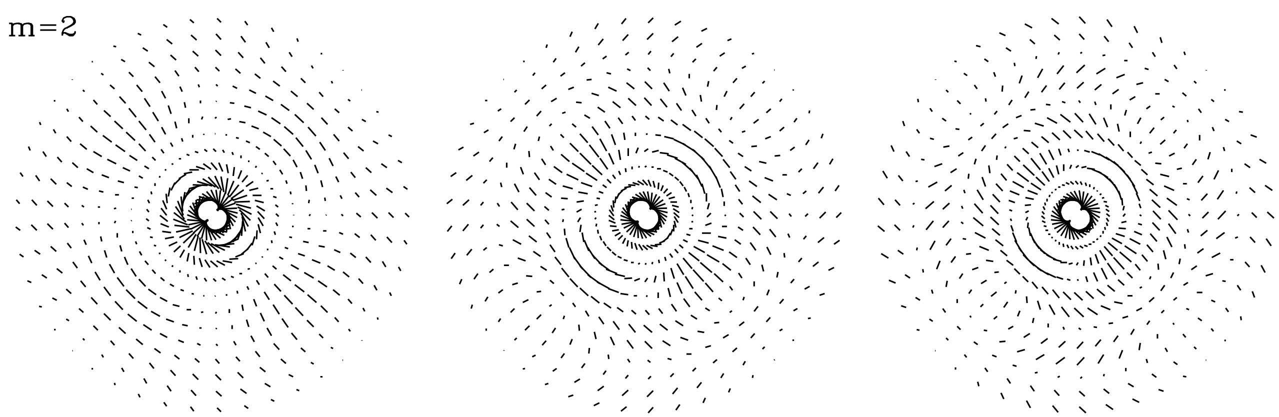
$$M_\perp(\vartheta) \equiv \int_0^\vartheta d\cos\theta \int_0^{2\pi} d\phi\, U(\theta,\phi) w(\theta;\vartheta)$$

Kaiser 1995
Schneider 1996

  - Can extend to get all $pure$ information for arbitrary geometries:

$$\int d^2\hat{\boldsymbol{n}}\, \mathcal{P}^{ab} Y^B_{ab}[\psi] \quad \text{where} \quad Y^B_{ab}[\psi] \equiv \epsilon^c{}_{(a}\nabla_{b)}\nabla_c\psi$$

    and ψ and $\nabla_a\psi$ *vanish on boundary*
  - N.b. polarization defined by $Y^B_{ab}[\psi]$ is at 45 deg to boundary
- $\langle M_\perp(\vartheta)^2\rangle$ usually estimated from ζ_{tt} and ζ_{rr} $\Rightarrow$ *retains* estimator-induced variance



*Lewis, AC & Turok 2002; Bunn et al. 2003

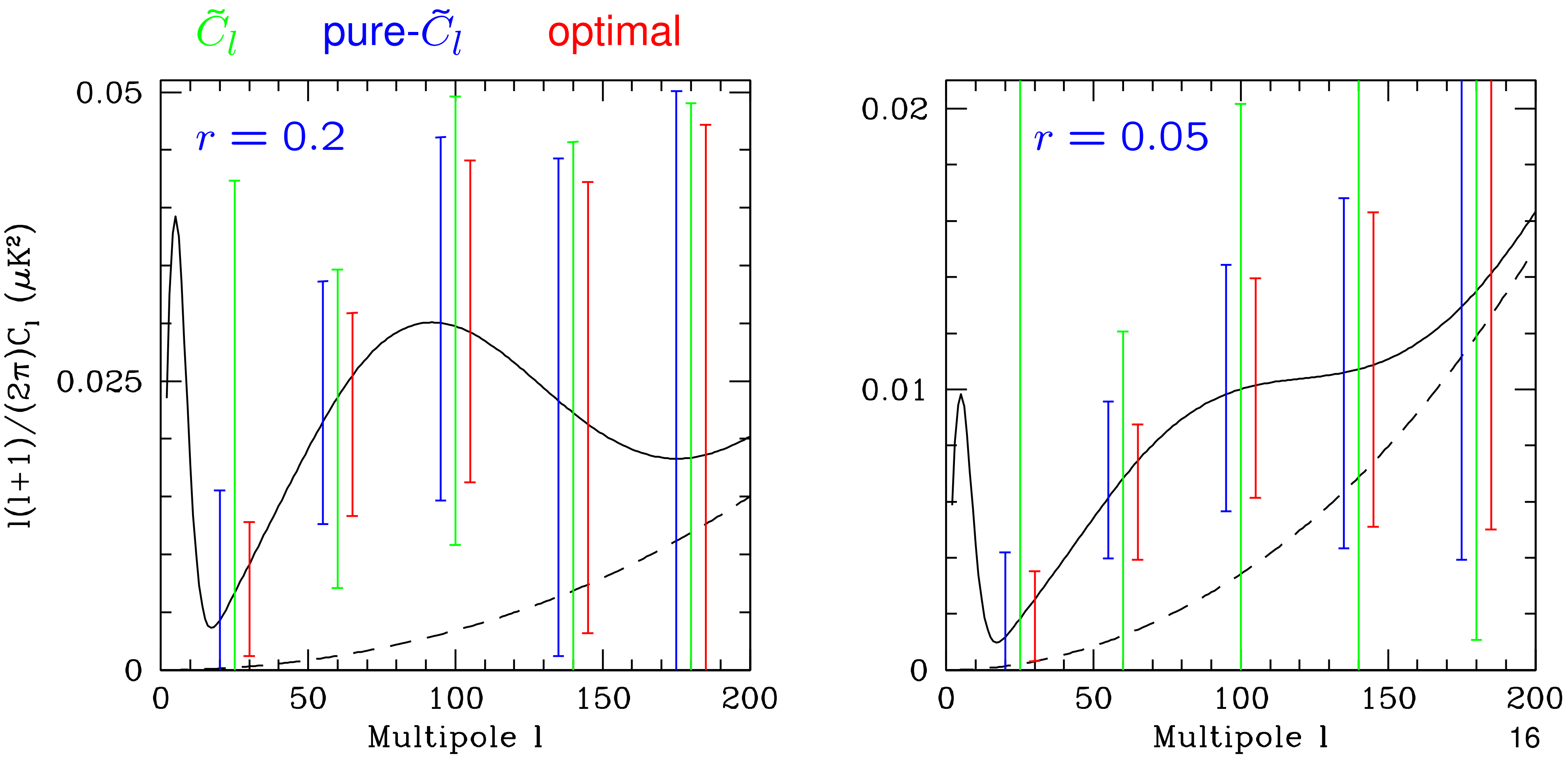
EFFICIENT ESTIMATION OF 'PURE-MODE' POWER SPECTRA

- In pC_l estimation, replace (Smith 2006)

$$\tilde{B}_{lm} \propto \int d^2\hat{\boldsymbol{n}}\, w(\hat{\boldsymbol{n}}) \mathcal{P}^{ab} Y^B_{ab}[Y_{lm}] \quad \text{with} \quad \int d^2\hat{\boldsymbol{n}}\, \mathcal{P}^{ab} Y^B_{ab}[w Y_{lm}]$$

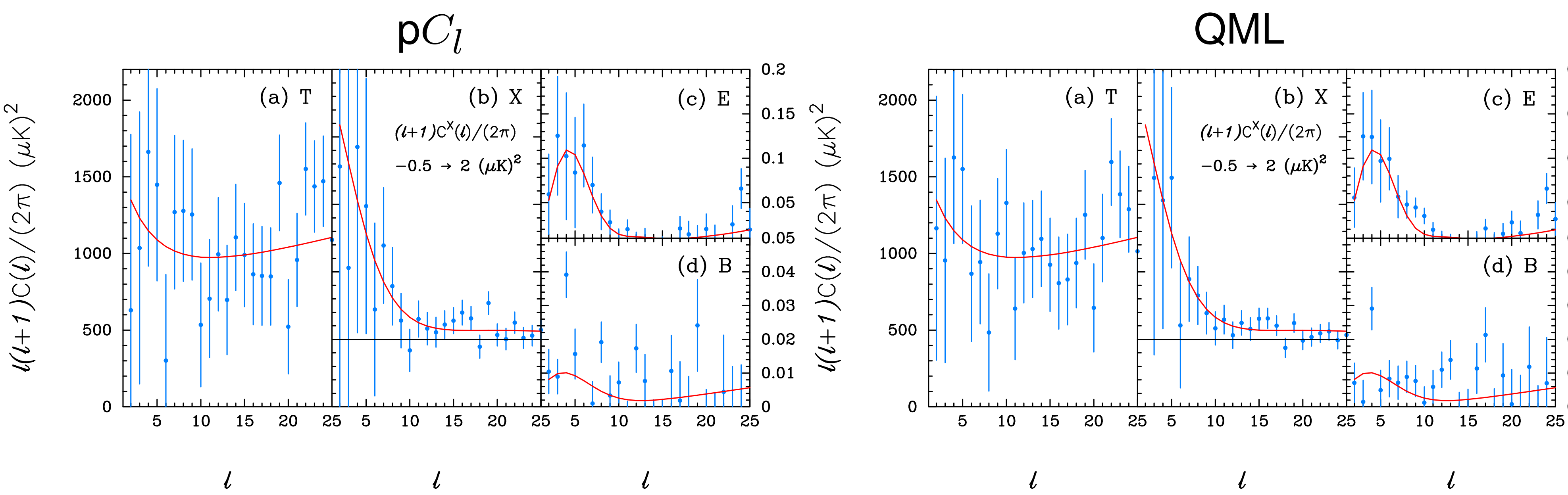
with $w = 0 = \nabla_a w$ on boundary for pure $\tilde{C}^B_l$

  – Retains speed of pC_l but eliminates excess variance from E-B mixing



EFFICIENT MAXIMUM-LIKELIHOOD ESTIMATION: HYBRID METHODS*

- M-L only beneficial on large scales when signal-dominated
  - Use QML on large scales and pC_l on small scales joining smoothly
- Smooth maps, repixelise and apply QML to coarse-pixel maps
  - Noise requires care since correlated in smoothed maps

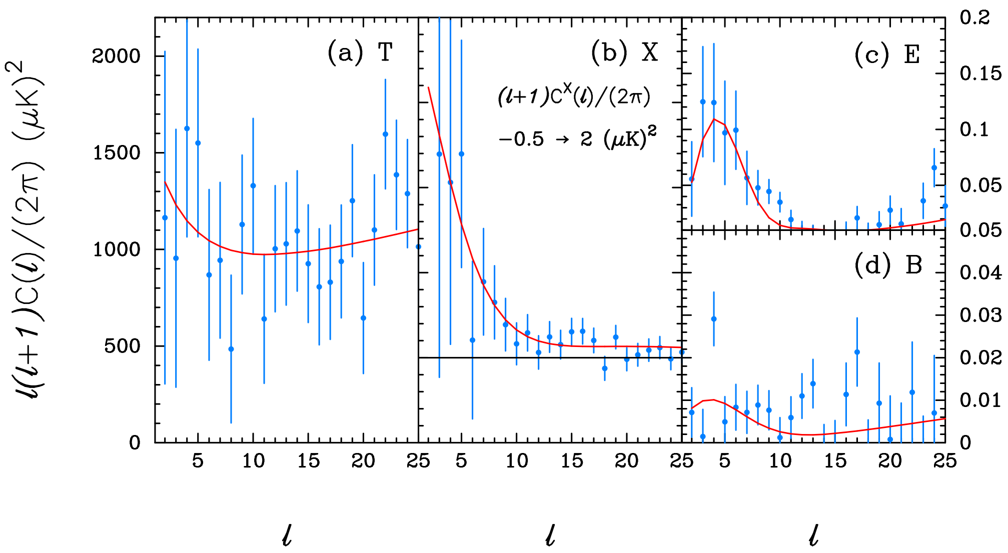


*Efstathiou (2004; 2006)

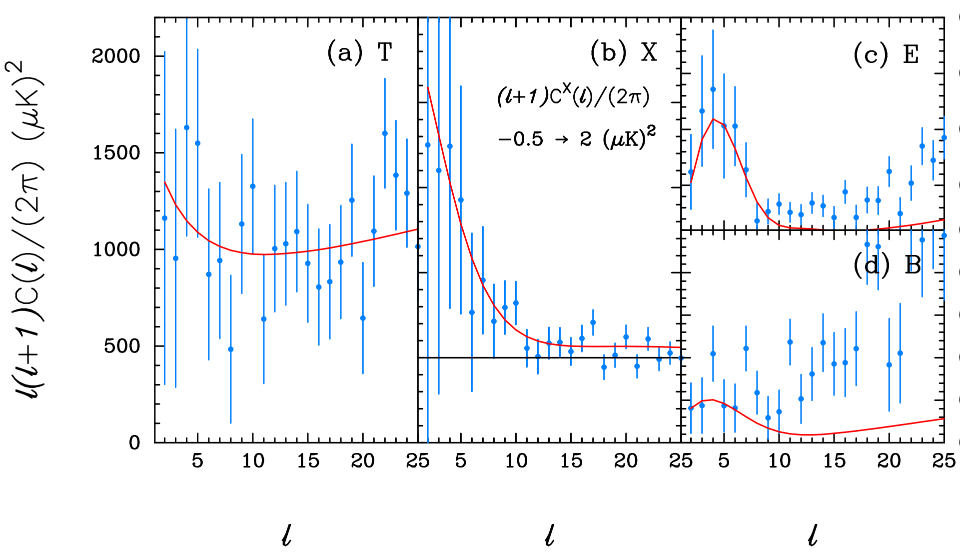
CORRELATED NOISE?

- Accurate noise bias subtraction critical for all estimators when noise-dominated
- Irreducible 'stripes' for Planck must be corrected for on large scales in EE and BB
- Analogous to B-modes from intrinsic alignments in cosmic shear

White noise



Correlated noise



Lessons learned from CMB polarization

- Interpretation of power spectrum simpler than correlation functions
  - E.g. uncorrelated bandpower measurements
  - But CMB naturally pixelised; not so cosmic shear
- Spherical sky methods (already developed for CMB) will be required for future large cosmic shear surveys
- Faster clustering analysis (pC_l-based correlation functions) may be worth considering for billion-galaxy surveys
- Worth adopting more careful weighting on large scales where sample variance dominant
- Scope for improved B-mode isolation on large scales $\Rightarrow$ tighter monitoring of systematics
- Error analysis more difficult for cosmic shear on small scales before shot noise dominates (c.f. non-Gaussian lens-induced B-modes for CMB)
- Considerable synergy but mostly disparate communities of analysts!
- No time: likelihood issues and full Bayesian approaches