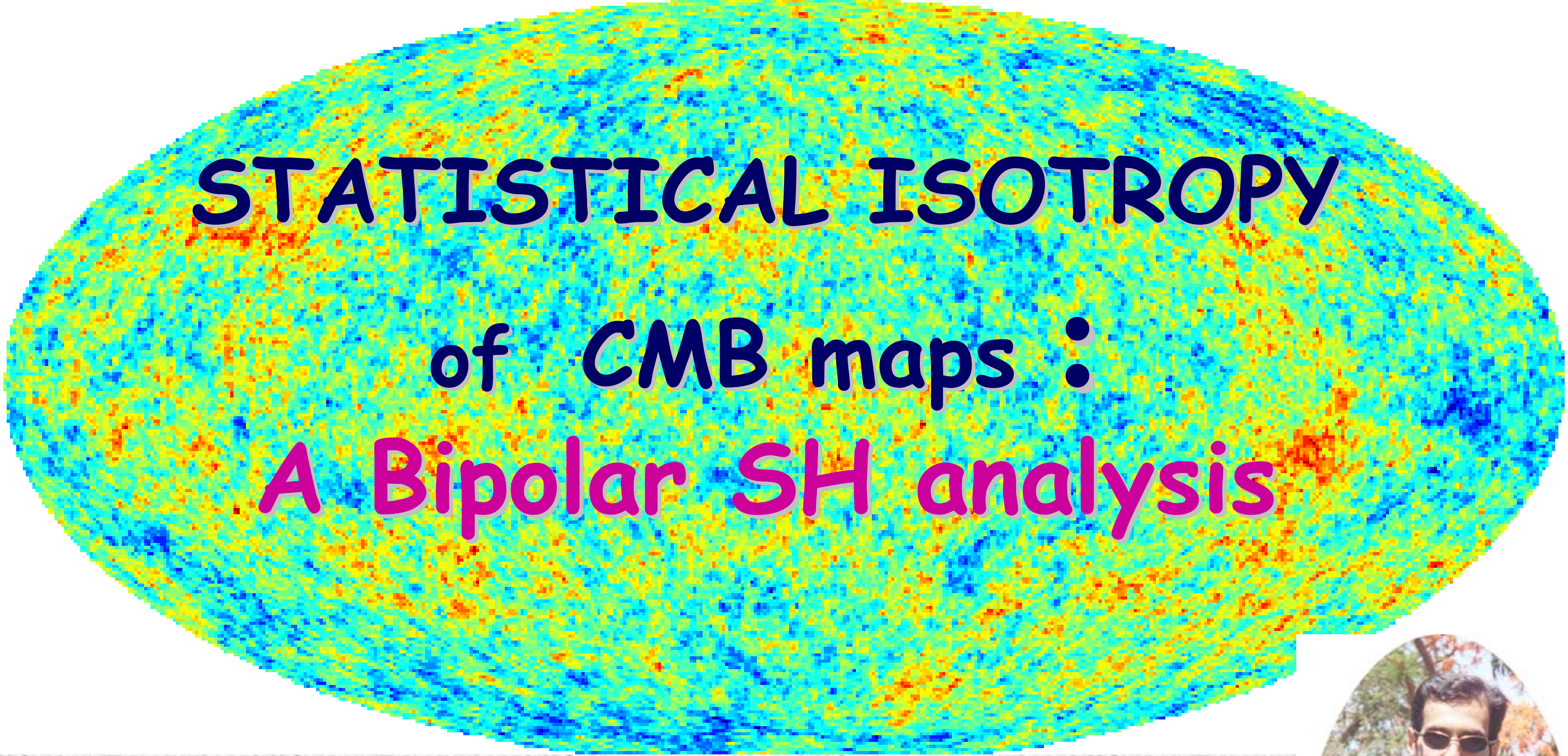
STATISTICAL ISOTROPY of CMB maps : A Bipolar SH analysis



20th. IAP Colloq.
(Jul. 2, 2004)

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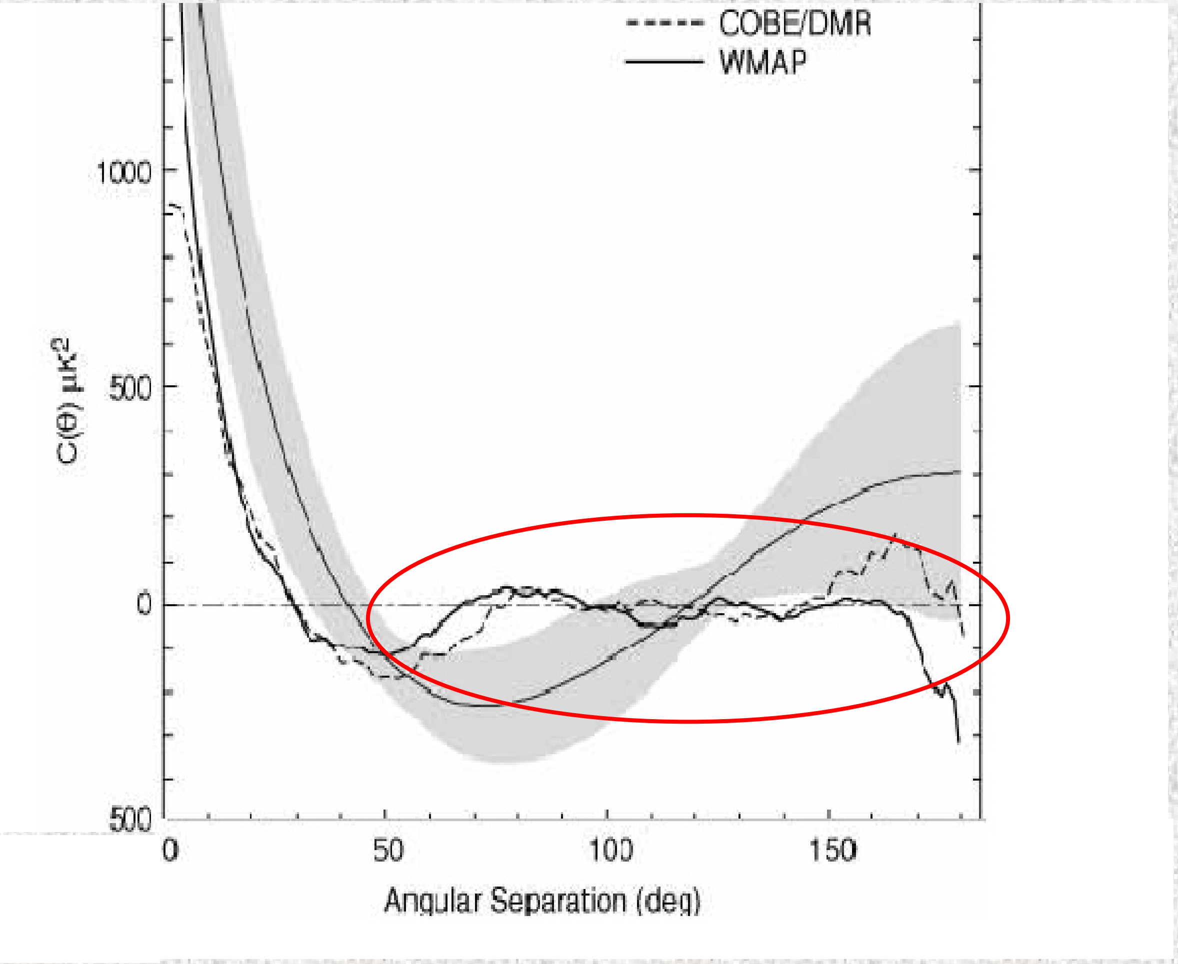
I.U.C.A.A, Pune

WMAP: Angular correlation function

Intriguing: Lack of power at large angular scales $(\theta \geq 60^o)$



Can imply more than just the suppression of power in the low multipoles !

NASA/WMAP science team

Asymmetries in the CMB anisotropy

<u>N-S asymmetry</u>

H. K. Eriksen, et al. 2004, F. K. Hansen et al. 2004a,b
(in local power)
Larson & Wandelt 2004, Park 2004
(genus stat.)

<u>Special directions</u>

Tegmark et al. 2004 (*l*=2,3 aligned)
Copi et al. 2004 (multipole vectors)
Land & Magueijo 2004 (cubic anomalies)
Prunet et al., 2004 (mode coupling)

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Broadly, stat. properties are not invariant under rotations

I.e., Breakdown of Statistical isotropy ?

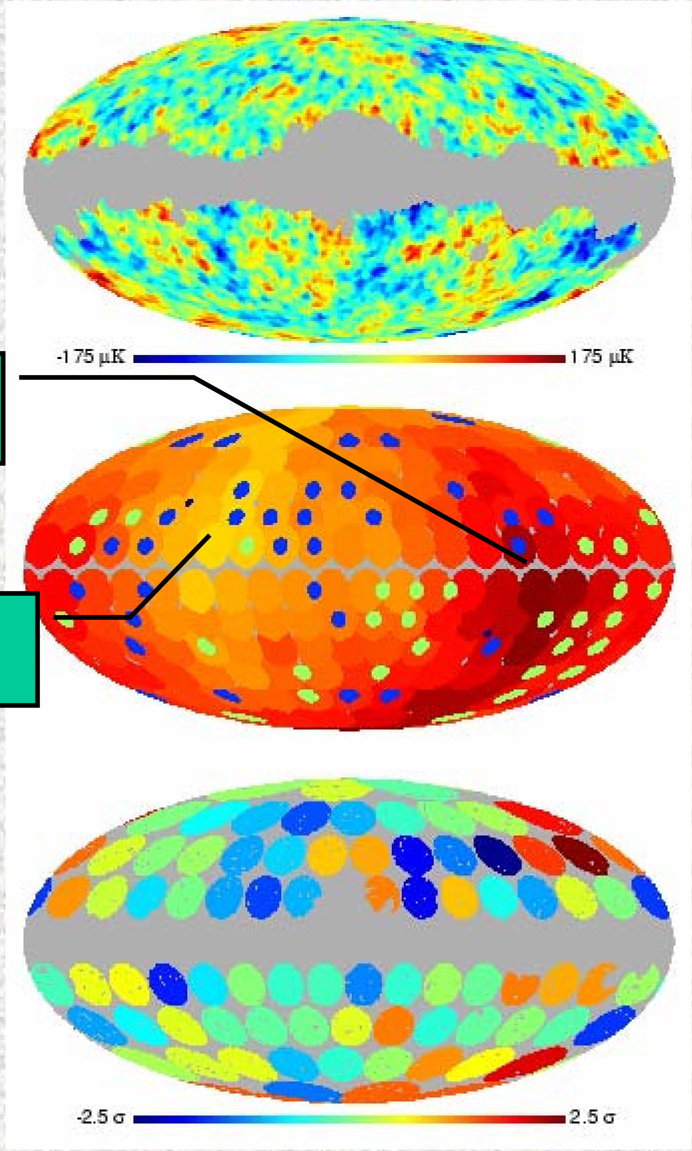


Fig: H. K. Eriksen, et al. 2003

Statistics of CMB

$\Delta T(\hat{n})$ smooth random function on a sphere (sky map).

General random CMB anisotropy: described by a

Probability Distribution Functional $P[\Delta T(\hat{n})]$

- Mean: $<\Delta T_i>=0$
- **Covariance** (2-point correlation) $C_{ij} \equiv C(\hat{n}_i, \hat{n}_j) = <\Delta T(\hat{n}_i)\Delta T(\hat{n}_j)>$
- ...

Gaussian Random CMB anisotropy

Completely specified by the **covariance matrix** C_{ij}

- N-point correlation $<\Delta T_i \Delta T_j ... \Delta T_N>$

Statistics of CMB

CMB anisotropy completely specified by the

angular power spectrum C_l

i.e., Correlation is invariant under rotations

Only if

$$C(\hat{n}_1, \hat{n}_2) \equiv C(\hat{n}_1 \bullet \hat{n}_2) = \frac{1}{4\pi} \sum_l (2l+1) C_l P_l(\hat{n}_1 \bullet \hat{n}_2)$$

Statistically isotropic Gaussian random CMB anisotropy

Iso-contours of correlation around a point $f(\hat{n}) \equiv C(\hat{n}, \hat{z})$

Radical breakdown of SI

disjoint iso-contours

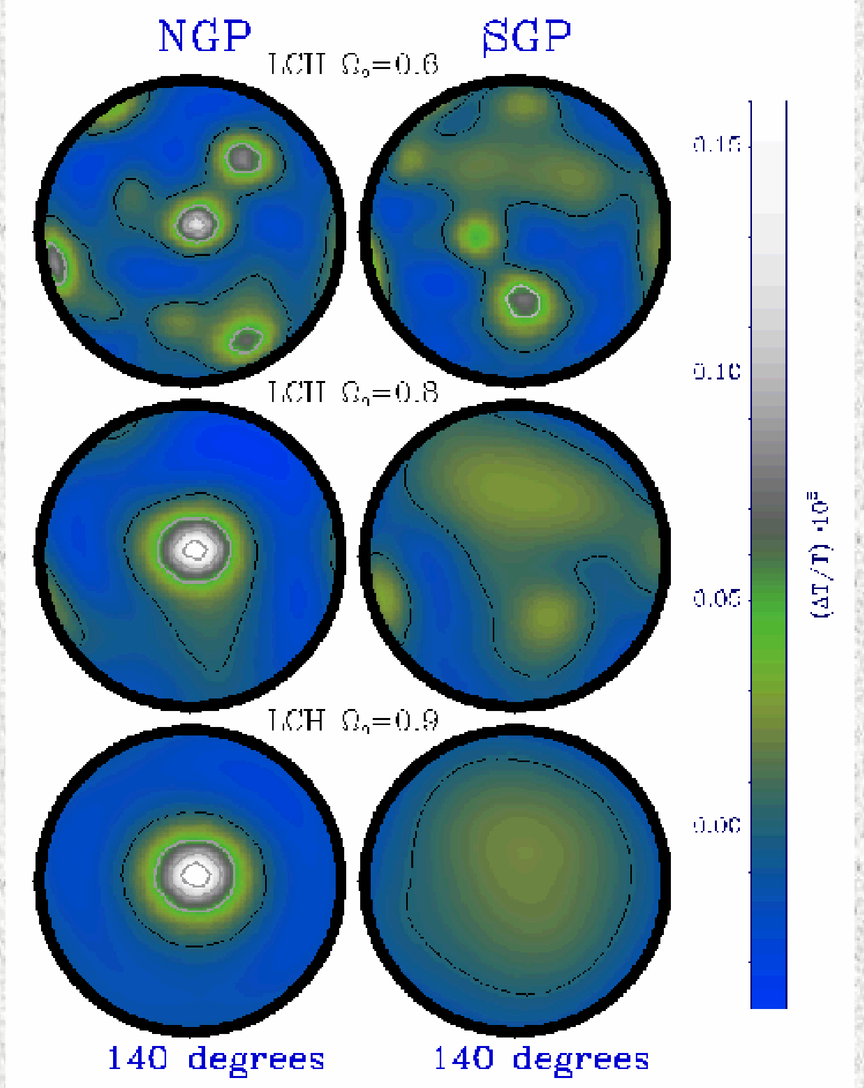
multiple imaging

Mild breakdown of SI

Distorted iso-contours

Statistically isotropic (SI)

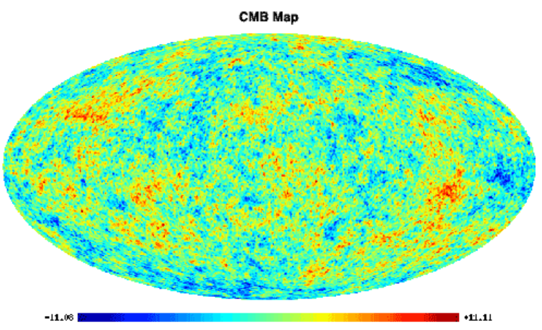
Circular iso-contours



(Bond, Pogosyan & Souradeep 1998, 2002)

Statistics of CMB

CMB Anisotropy Sky map => Spherical Harmonic decomposition

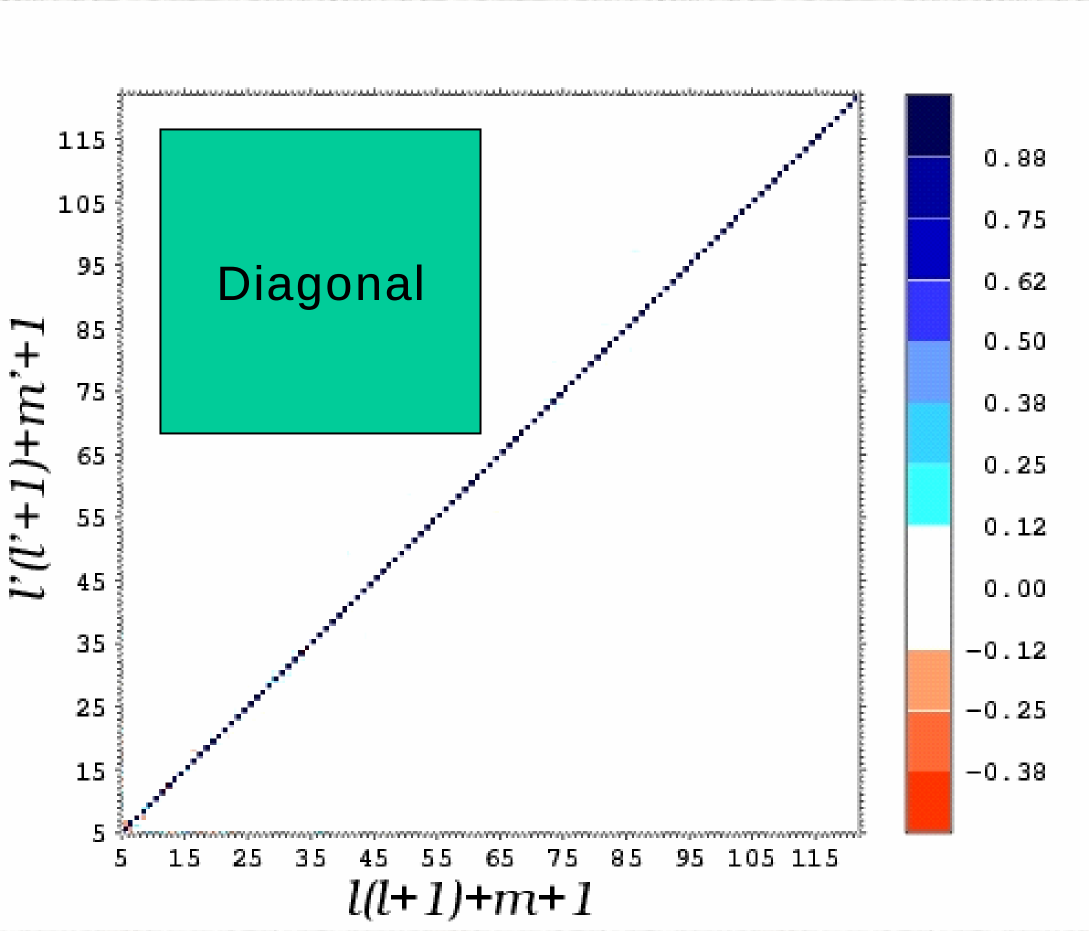


$$\Delta T(\theta,\phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi)$$

Statistical isotropy

$$\left\langle a_{lm} a^{*}_{l'm'} \right\rangle = C_l \; \delta_{ll'} \delta_{mm'}$$

Single index n:
(l,m) -> n

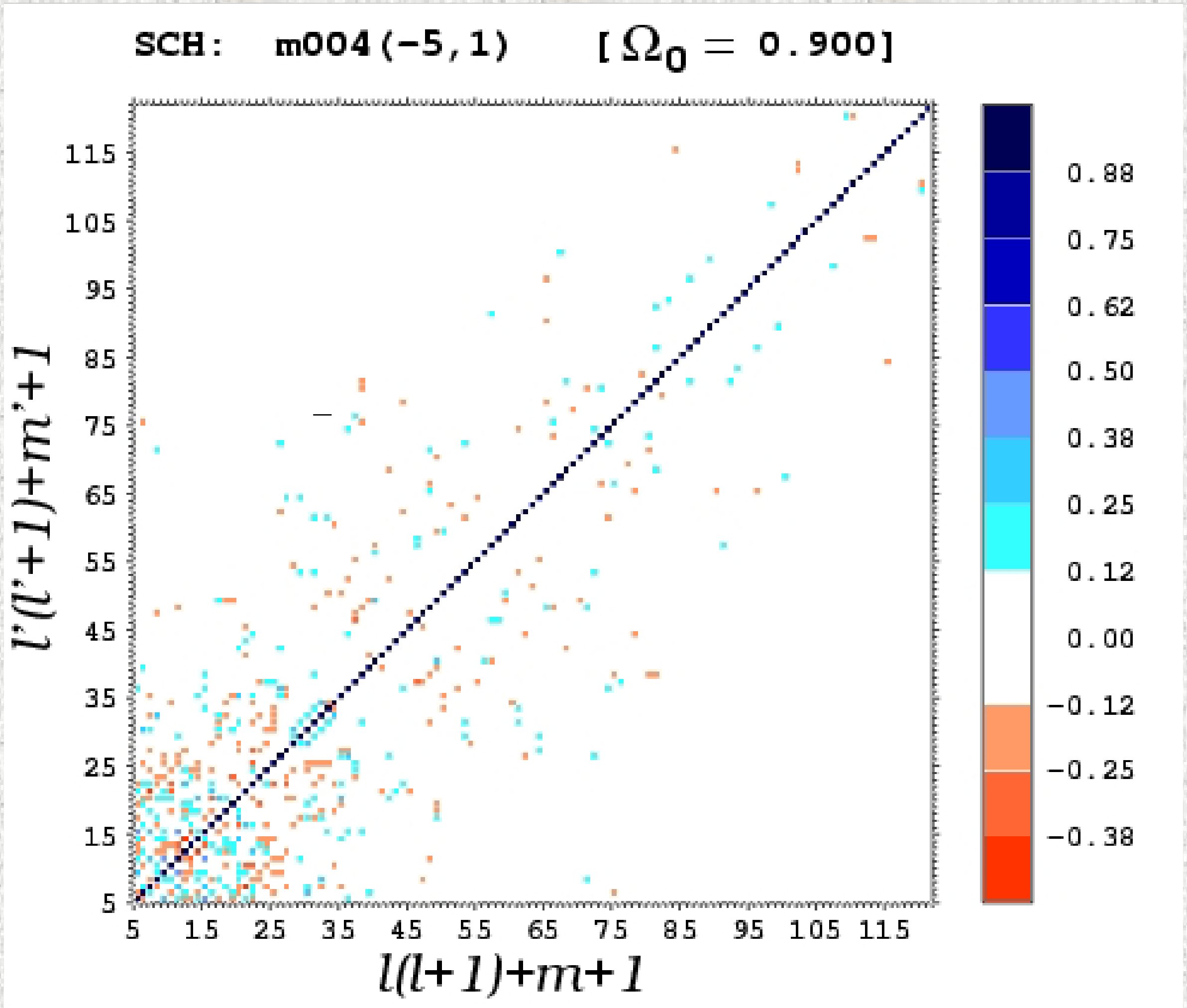


$$\langle a_{lm} a^{*}_{l'm'} \rangle = C_l \, \delta_{ll'} \delta_{mm'}$$

SI violation : $\langle a_{lm} a^*_{l'm'} \rangle \neq C_l \; \delta_{ll'} \delta_{mm'}$

Mild breakdown

$$\frac{\langle a_{lm} a^*_{l'm'} \rangle}{\sqrt{\langle a_{l'm'} a^*_{l'm'} \rangle \langle a_{lm} a^*_{lm} \rangle}}$$

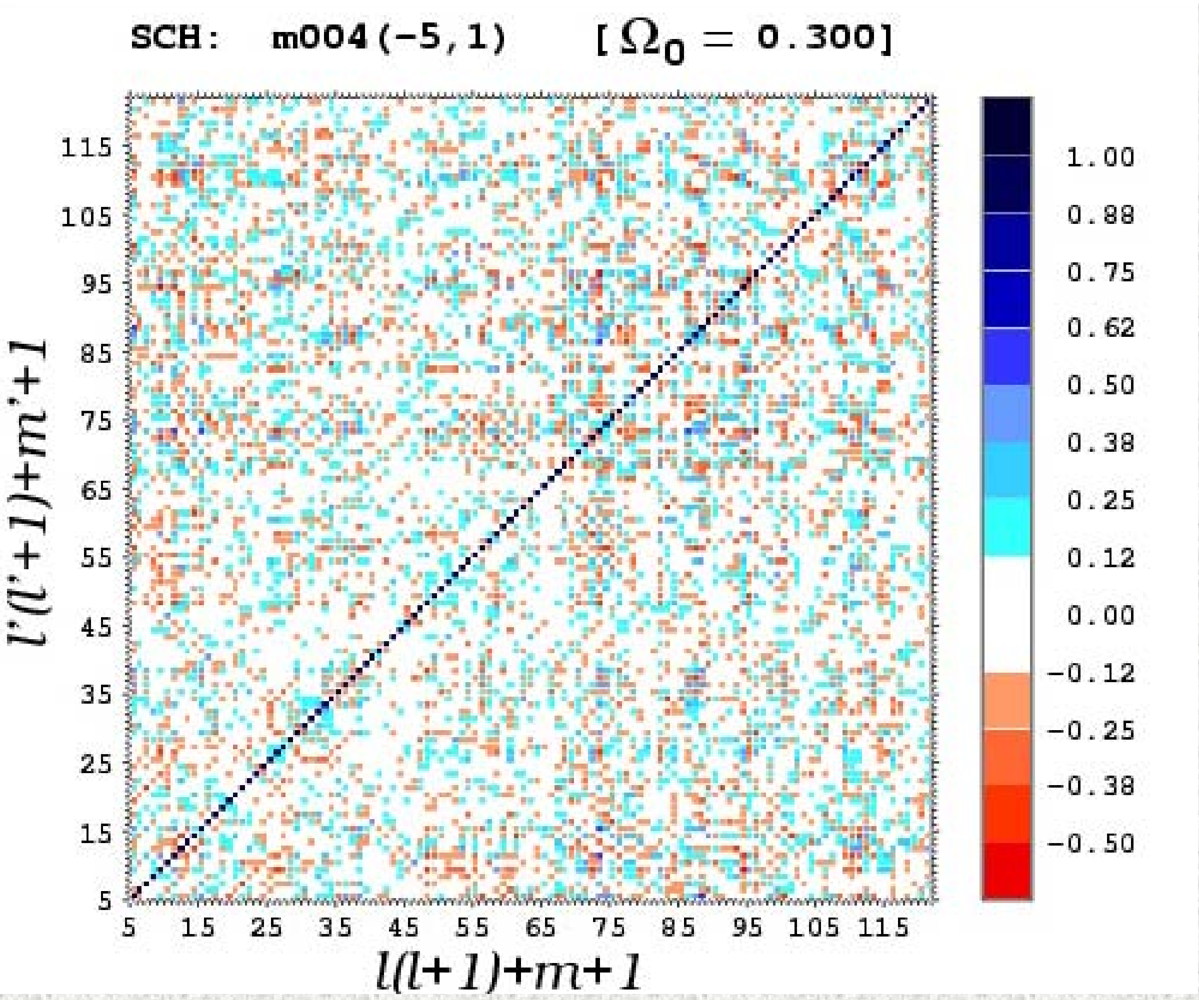


(Bond, Pogosyan & Souradeep 1998, 2002)

SI violation : $\langle a_{lm} a^*_{l'm'} \rangle \neq C_l \; \delta_{ll'} \delta_{mm'}$

Radical breakdown

$$\frac{\langle a_{lm} a^*_{l'm'} \rangle}{\sqrt{\langle a_{l'm'} a^*_{l'm'} \rangle \langle a_{lm} a^*_{lm} \rangle}}$$



(Bond, Pogosyan & Souradeep 1998, 2002)

SI violation, or ... Correlation patterns

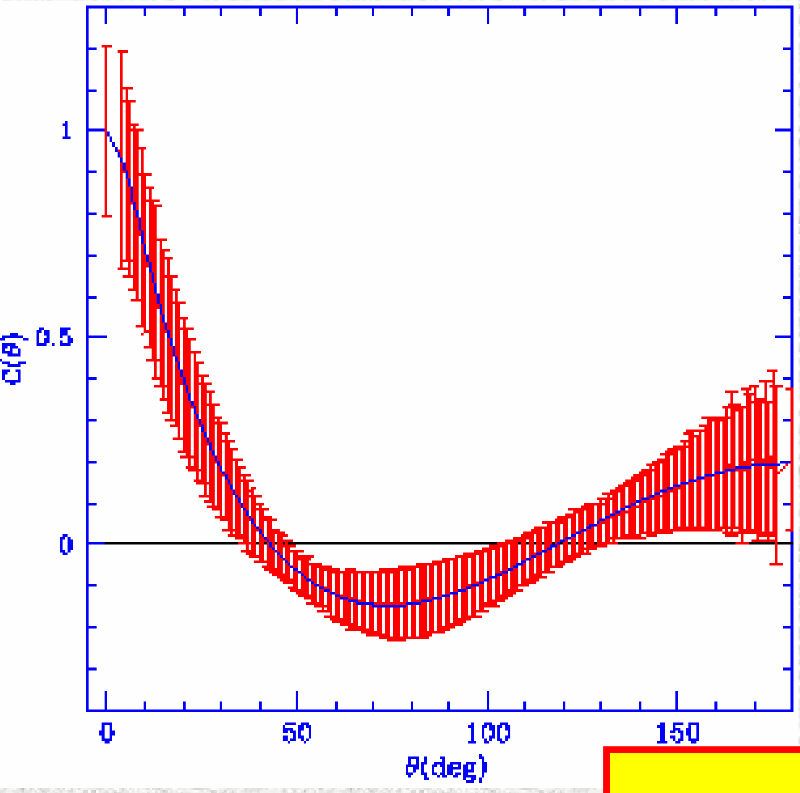
Beautiful Correlation patterns could underlie the CMB tapestry

Can we Measure Correlation Patterns?

the *COSMIC CATCH* is

there is only one CMB sky !

Measuring the SI correlation



Statistical isotropy

$C(\theta)$ can be well estimated by averaging over the temperature product between all pixel pairs separated by an angle θ.

$$\tilde{C}(\theta) = \sum_{\hat{n}_1}\sum_{\hat{n}_2} \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\delta(\hat{n}_1 \cdot \hat{n}_2 - \cos\theta)$$

$$C(\hat{n}_1 \bullet \hat{n}_2) = \frac{1}{8\pi^2}\int d\Re \; C(\Re\hat{n}_1, \Re\hat{n}_2)$$

Measuring the non-SI correlation

In the absence of statistical isotropy

Estimate of the correlation function from a sky map given by a single temperature product $\tilde{C}(\hat{n}_1, \hat{n}_2) = \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)$

is poorly determined!!

(unless it is a KNOWN pattern)

- **Matched circles statistics** (Cornish, Starkman, Spergel '98)
- **Anticorrelated ISW circle centers** (Bond, Pogosyan, TS '98, '02)
- **Planar reflective symmetries** (de OliveiraCosta, Smoot Starobinsky '96)

Known correlation →Full Bayesian Analysis

Compact universes

COBE data :Bond, Pogosyan & TS 1998, 2002

WMAP data : Phillips & Kogut 2004, Pogosyan et al. 04

Given data $\left\{\Delta T_i^d\right\}$, and an estimate of the Noise matrix

Probability of any model M : $C_S(\{p_i\})$

$$P\left[M \mid \left\{\Delta T_i^d\right\}\right] \propto P\left[\left\{\Delta T_i^d\right\} \mid M\right] \quad : \text{Bayes Thm.}$$

$$= \frac{1}{\sqrt{(2\pi)^{N_p} \det(C)}} \exp - \left[\frac{1}{2} \sum_{ij} \Delta T_i^d \; C_{ij}^{-1} \; \Delta T_j^d\right]$$

D. Pogosyan's talk

Bipolar Power spectrum (BiPS) :

A Generic Measure of Statistical Anisotropy

$$\text{Recall}: C(\hat{n}_1 \bullet \hat{n}_2) = \frac{1}{8\pi^2}\int d\Re \; C(\Re\hat{n}_1, \Re\hat{n}_2)$$

Bipolar multipole index

$$\kappa^{\ell} = \int d\Omega_{n_1} \int d\Omega_{n_2} \left[\frac{1}{8\pi^2}\int d\Re \; \chi^{\ell}(\Re)\, C(\Re\hat{n}_1, \Re\hat{n}_2)\right]^2$$

A **weighted average** of the correlation function over all rotations

$$\chi^{\ell}(\Re) = \sum_{m=-\ell}^{\ell} D^{\ell}_{mm}(\Re)$$

Characteristic function

Wigner rotation matrix

Statistical Isotropy $\Rightarrow \kappa^{\ell} = \kappa^{0}\delta_{\ell 0}$

Correlation is invariant under rotations

$$C(\Re\hat{n}_1, \Re\hat{n}_2) = C(\hat{n}_1, \hat{n}_2)$$

$$\kappa^{\ell} = (2\ell + 1)^2 \int d\Omega_{n_1} \int d\Omega_{n_2} C^2(\hat{n}_1, \hat{n}_2)[\frac{1}{8\pi^2} \int d\Re \chi^{\ell}(\Re)]^2$$

$$\int d\Re \chi^{\ell}(\Re) = \delta_{\ell 0}$$

BiPS: In Harmonic Space

- Correlation is a *two point function* on a sphere

$$C(\hat{n}_1,\hat{n}_2) = \sum_{l_1 l_2 LM} A^{LM}_{l_1 l_2} \{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\}_{LM}$$

BiPoSH

Bipolar spherical harmonics.

$$\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\}_{LM} = \sum_{m_1 m_2} C^{LM}_{l_1 l_2 m_1 m_2} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2)$$

Clebsch-Gordan

- Inverse-transform

$$A^{LM}_{l_1 l_2} = \int d\Omega_{n_1} \int d\Omega_{n_2} C(\hat{n}_1,\hat{n}_2) \{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\}^{*}_{LM}$$

$$= \sum_{m_1 m_2} \langle a_{l_1 m_1} a_{l_2 m_2} \rangle \; C^{LM}_{l_1 m_1 l_2 m_2}$$

Linear combination of off-diagonal elements

Recall: Coupling of angular momentum states

$\langle l_1 m_1 l_2 m_2 \,|\, \ell M \rangle \qquad |l_1 - \ell| \le l_2 \le l_1 + \ell, \quad m_1 + m_2 + M = 0$

BiPoSH coefficients :

$$A^{\ell M}_{l_1 l_2} = \sum_{m_1} \left\langle a_{l_1 m_1} a^{*}_{l_2\; M+m_1} \right\rangle \; C^{\ell M}_{l_1 m_1 l_2\; M+m_1}$$

- Complete,Independent linear combinations of off-diagonal correlations.
- Encompasses other specific measures of off-diagonal terms, such as
  - Durrer e $D_l \equiv \left\langle a_{lm} a_{l+2\;\, m} \right\rangle = \sum_{\ell M} A^{\ell M}_{ll'} C^{\ell M}_{l+2\;\, m\;\, l\;\, m}$
  - Prunet et al. '04 : $D_l^{(i)} \equiv \left\langle a_{lm} a_{l+1\;\, m+i} \right\rangle = \sum_{\ell M} A^{\ell M}_{ll'} C^{\ell M}_{l+1\;\, m+i\;\, l\;\, m}$

BiPS:

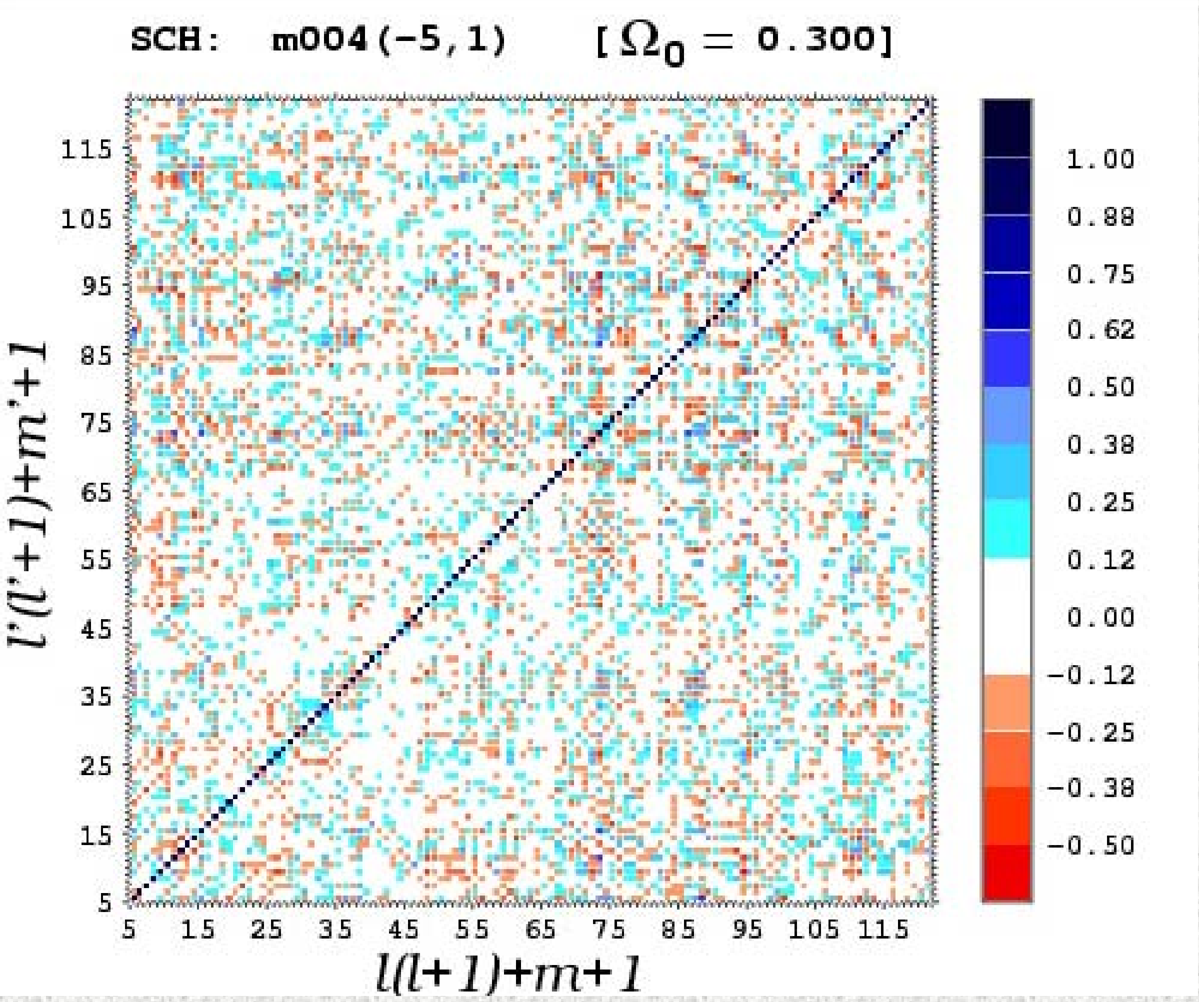
rotationally invariant

$$\kappa^{\ell} \equiv \sum_{M, l_1, l_2} | A^{\ell M}_{l_1 l_2} |^2 \ge 0$$

SI violation : $\langle a_{lm} a^*_{l'm'} \rangle \neq C_l \; \delta_{ll'} \delta_{mm'}$

Radical breakdown

$$\frac{\langle a_{lm} a^*_{l'm'} \rangle}{\sqrt{\langle a_{l'm'} a^*_{l'm'} \rangle \langle a_{lm} a^*_{lm} \rangle}}$$



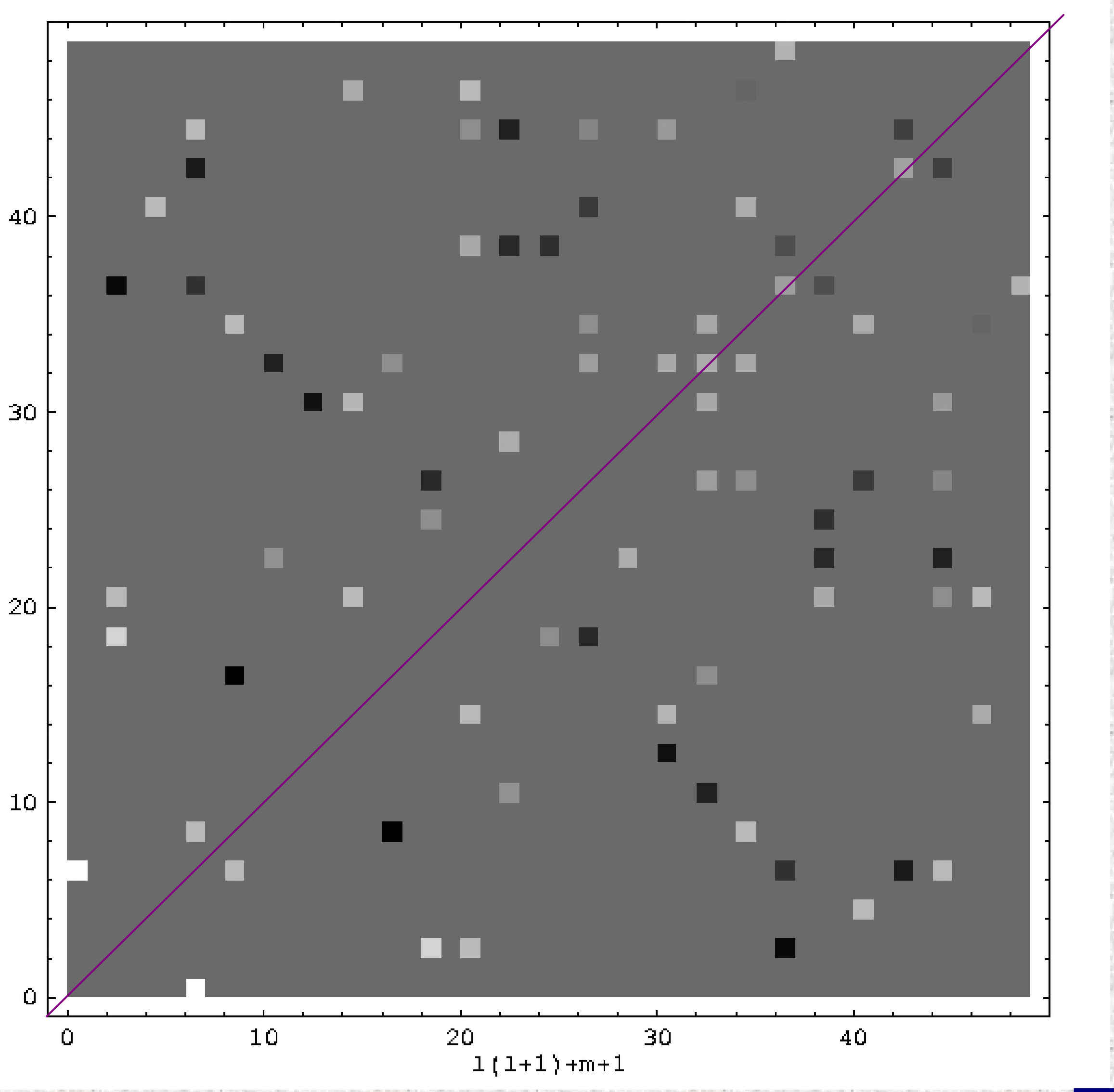
(Bond, Pogosyan & Souradeep 1998, 2002)

Structure of BiPoSH

$$\kappa_2 = \sum_{ll'M} \left| A^{2M}_{ll'} \right|^2$$

$$A^{20}_{ll'}$$

$(l', m') \rightarrow n'$



$(l, m) \rightarrow n = l(l+1) + m + 1$

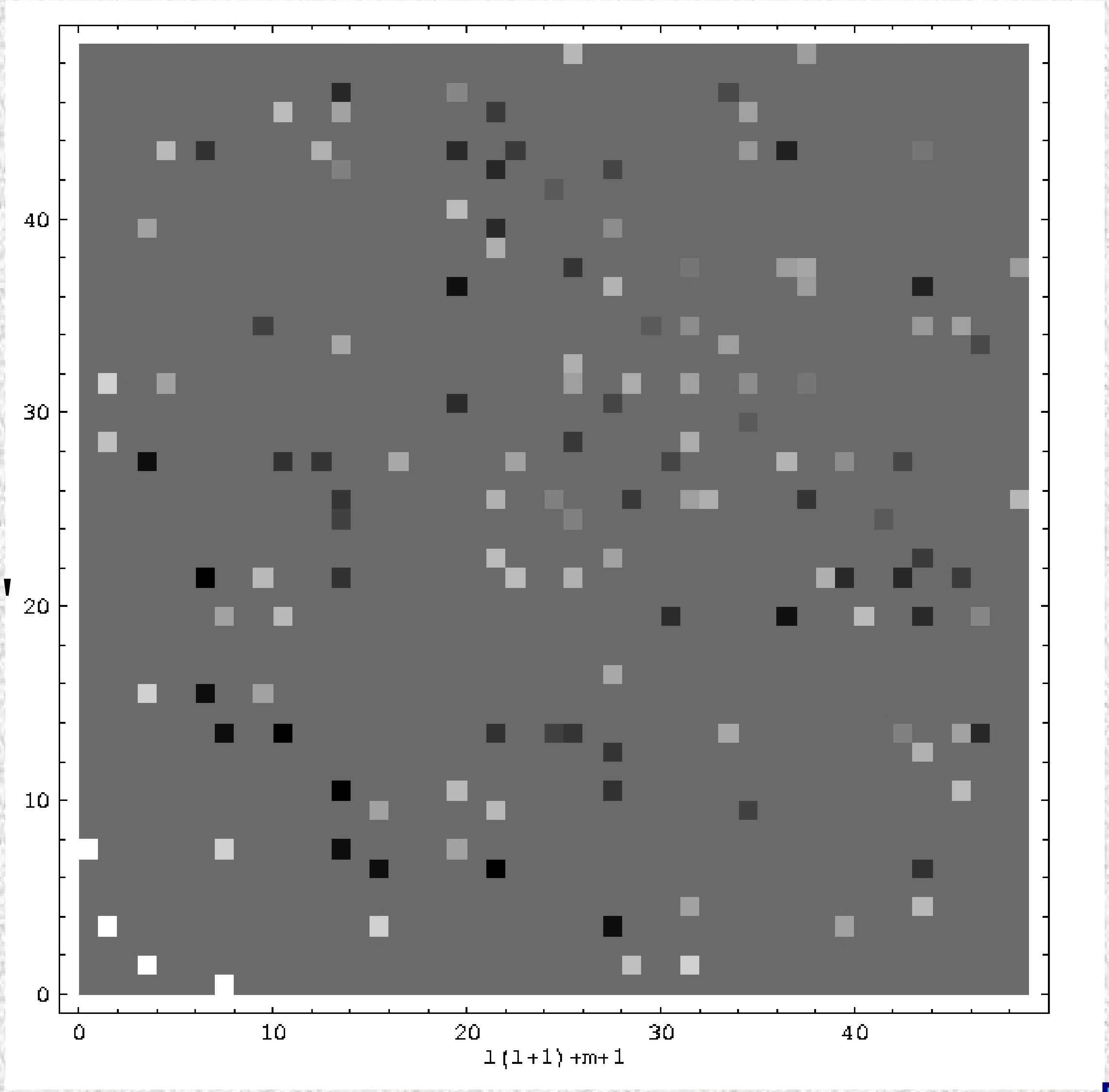
$(l_{max} = 6)$

Structure of BiPoSH

$$\kappa_2 = \sum_{ll'M} \left| A^{2M}_{ll'} \right|^2$$

$$A^{21}_{ll'}$$

$$(l', m') \to n'$$



$$(l, m) \to n = l(l+1) + m + 1$$

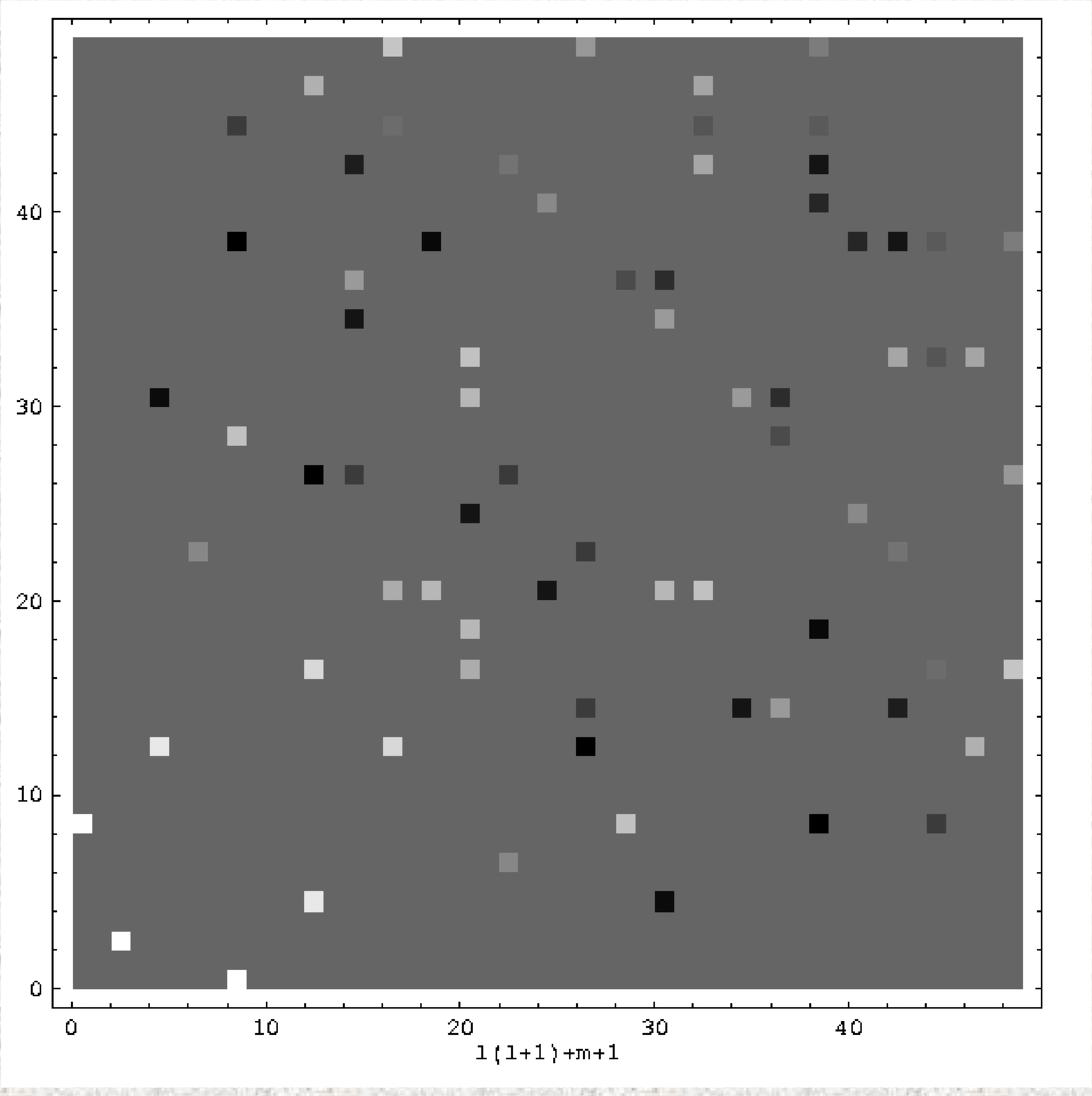
$$(l_{\max} = 6)$$

Structure of BiPoSH

$$\kappa_2 = \sum_{ll'M} \left| A_{ll'}^{2M} \right|^2$$

$$A_{ll'}^{22}$$

$$(l', m') \rightarrow n'$$



$$(l, m) \rightarrow n = l(l+1) + m + 1$$

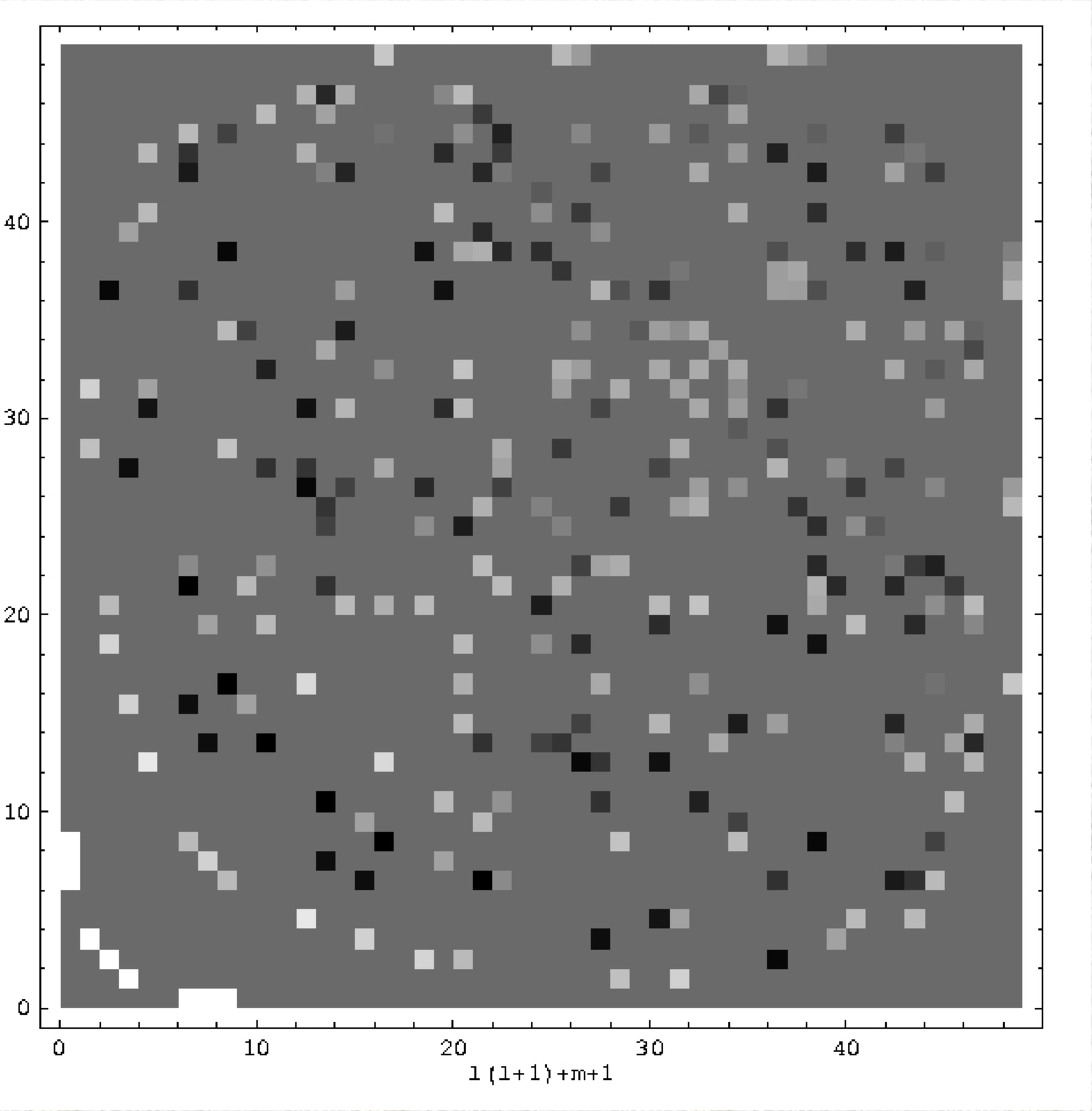
$$(l_{\max} = 6)$$

Structure of BiPoSH

$$\kappa_2 = \sum_{ll'M} \left| A^{2M}_{ll'} \right|^2$$

$$\sum_M A^{2M}_{ll'}$$

$$(l', m') \rightarrow n'$$



$$(l, m) \rightarrow n = l(l+1) + m + 1$$

$$(l_{max} = 6)$$

Spherical harmonics	**Bipolar spherical harmonics**
a_{lm}	$A^{\ell M}_{ll'}$
Spherical Harmonic coefficents	BiPoSH coefficents
C_l	κ^{ℓ}
Angular power spectrum	**BiPS**

Spherical harmonics	**Bipolar spherical harmonics**
a_{lm}	$A^{\ell M}_{ll'}$
Spherical Harmonic Transforms	BipoSH Transforms
C_l	κ^{ℓ}
Angular power spectrum	**BiPS**

Measure of Statistical Isotropy

$$A^{\ell M}_{ll'} = \sum_{mm'} a_{lm} a_{l'm'} \; C^{\ell M}_{lml'm'}$$

SH transform of the map

$$\kappa_\ell = \sum_{ll'M} \left| A^{\ell M}_{ll'} \right|^2 - B_\ell$$

bias

$$\text{Stat. isotropy} \Rightarrow \kappa^\ell = \kappa^0 \delta_{\ell 0}$$

- Averaging over l,l'& M beats down Cosmic variance .
- Fast: Advantage of fast SH transform.

(8 mins /alpha 1.25 GHz proc.: Healpix 512, BiPS upto 20)

- Orientation independent.

Cosmic Bias

$$B_\ell = \langle \tilde{\kappa}_\ell \rangle - \langle \kappa_\ell \rangle$$

- Analytically calculate multi-D integrals over

$$< \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\Delta T(\hat{n}_3)\Delta T(\hat{n}_4) >$$

  – Gaussian statistics => express as products of covariance

For SI correlation

$$B_\ell = (2\ell+1)\sum_{l_1=2}^{\infty}\sum_{l_2=|l_1-\ell|}^{|l_1+\ell|} C_{l_1} C_{l_2} (1+(-1)^\ell \delta_{l_1 l_2})$$

"True" Cl

(A. Hajian and Souradeep, ApJ Lett. 2003)

Cosmic Variance

$$(\Delta\kappa_\ell)^2 = \left\langle \tilde{\kappa}_\ell^{\,2} \right\rangle - \left\langle \tilde{\kappa}_\ell \right\rangle^2$$

- Analytically calculate multi-D integrals over

$$< \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\Delta T(\hat{n}_3)\Delta T(\hat{n}_4)\Delta T(\hat{n}_5)\Delta T(\hat{n}_6)\Delta T(\hat{n}_7)\Delta T(\hat{n}_8) >$$

  - **Gaussian statistics => express as products of covariance.**

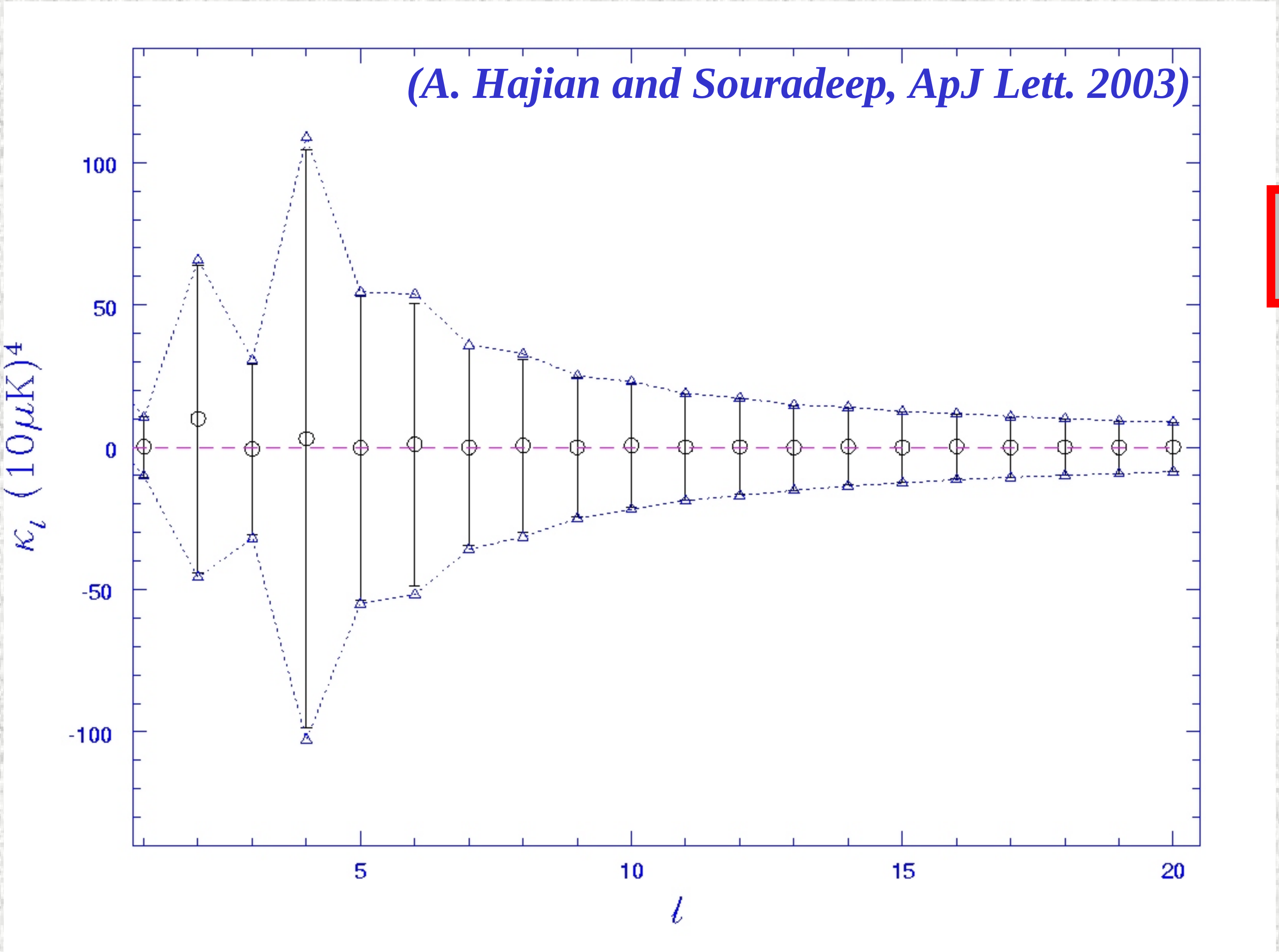
    Tedious exercise: 105 terms, 96 connected terms.

$$\begin{aligned}\mathrm{var}(\kappa_\ell) = &\sum_{l_1} C_{l_1}^4 \left(9\frac{(2\ell+1)^2}{2l_1+1} + 4(-1)^\ell(2\ell+1)\right) + 4(2\ell+1)\sum_{l_1,l_2} C_{l_1}^2 C_{l_2}^2 + 15(-1)^\ell \sum_{l_1,l_2} \frac{(2\ell+1)^2}{2l_1+1} C_{l_1}^3 C_{l_2} \\ &+ 8\sum_{l_1,l_2,l_3} \frac{(2\ell+1)^2}{2l_1+1} C_{l_1}^2 C_{l_2} C_{l_3} + 4(2+(-1)^\ell)\sum_{l_1} C_{l_1}^4 \sum_{M,M'} \sum_{m_i=-l_1}^{l_1} C^{\ell M}_{l_1 -m_1 l_1 -m_2} C^{\ell M}_{l_1 m_3 l_1 m_4} C^{\ell M'}_{l_1 m_2 l_1 m_4} C^{\ell M'}_{l_1 -m_1 l_1 -m_3} \\ &+ 4\sum_{l_1,l_2} C_{l_1}^2 C_{l_2}^2 \sum_{M,M'} \sum_{m_1,m_3=-l_1}^{l_1} \sum_{m_2,m_4=-l_2}^{l_2} C^{\ell M}_{l_1 -m_1 l_2 -m_2} C^{\ell M}_{l_1 m_3 l_2 m_4} C^{\ell M'}_{l_2 m_4 l_1 m_1} C^{\ell M'}_{l_2 -m_2 l_1 -m_3}\end{aligned}$$

"True" underlying theory

(A. Hajian and Souradeep, ApJ Lett. 2003)

Bias corrected BiPS measurement



Bias

$$B_\ell = \langle \tilde{\kappa}_\ell \rangle - \langle \kappa_\ell \rangle$$

Cosmic Variance

$$(\Delta\kappa_\ell)^2 = \langle \tilde{\kappa}_\ell^{\,2} \rangle - \langle \tilde{\kappa}_\ell \rangle^2$$

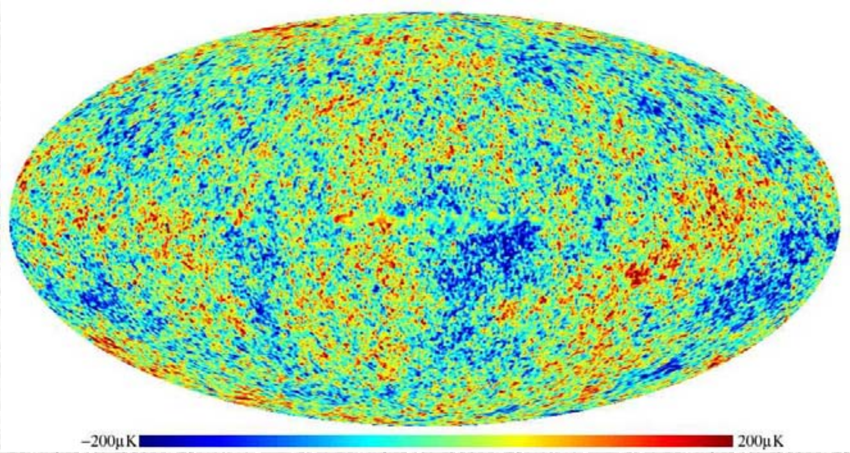
$$\Delta\kappa_\ell \propto \frac{1}{\ell}$$

Analytic estimate for bias and cosmic variance match numerical measurements on simulated statistically isotropic maps !

Testing Statistical Isotropy of WMAP

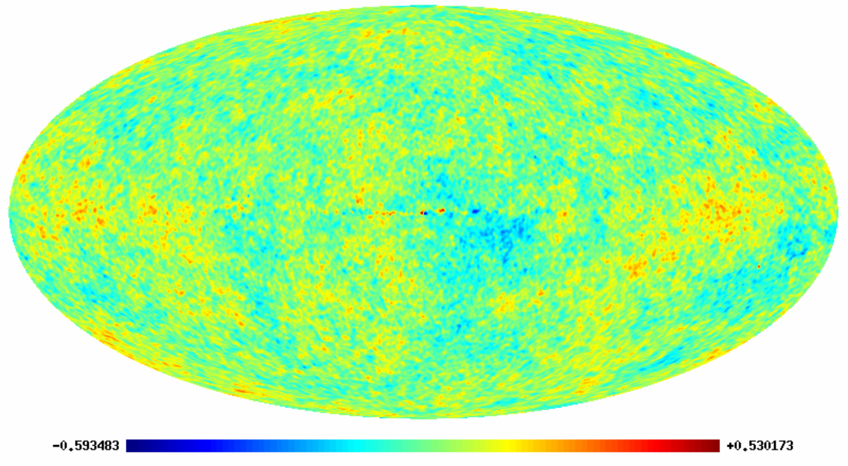
(for WMAP best fit model)

(Hajian, TS, Cornish astro-ph/0406354)



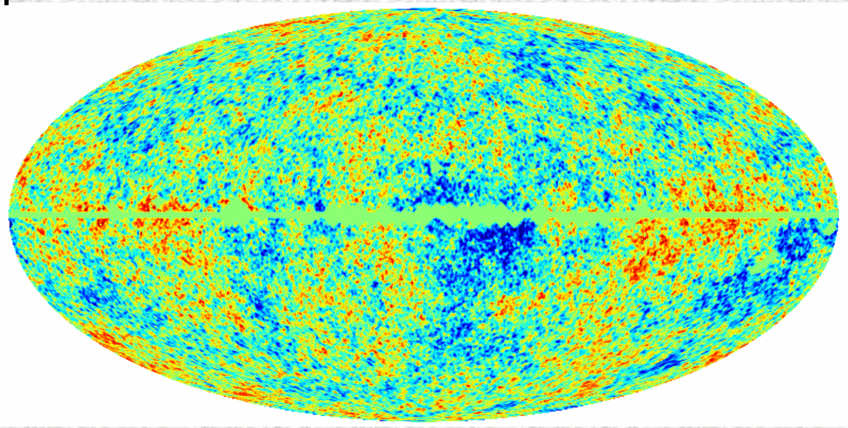
Foreground cleaned map

(Tegmark et al. 2003)



ILC

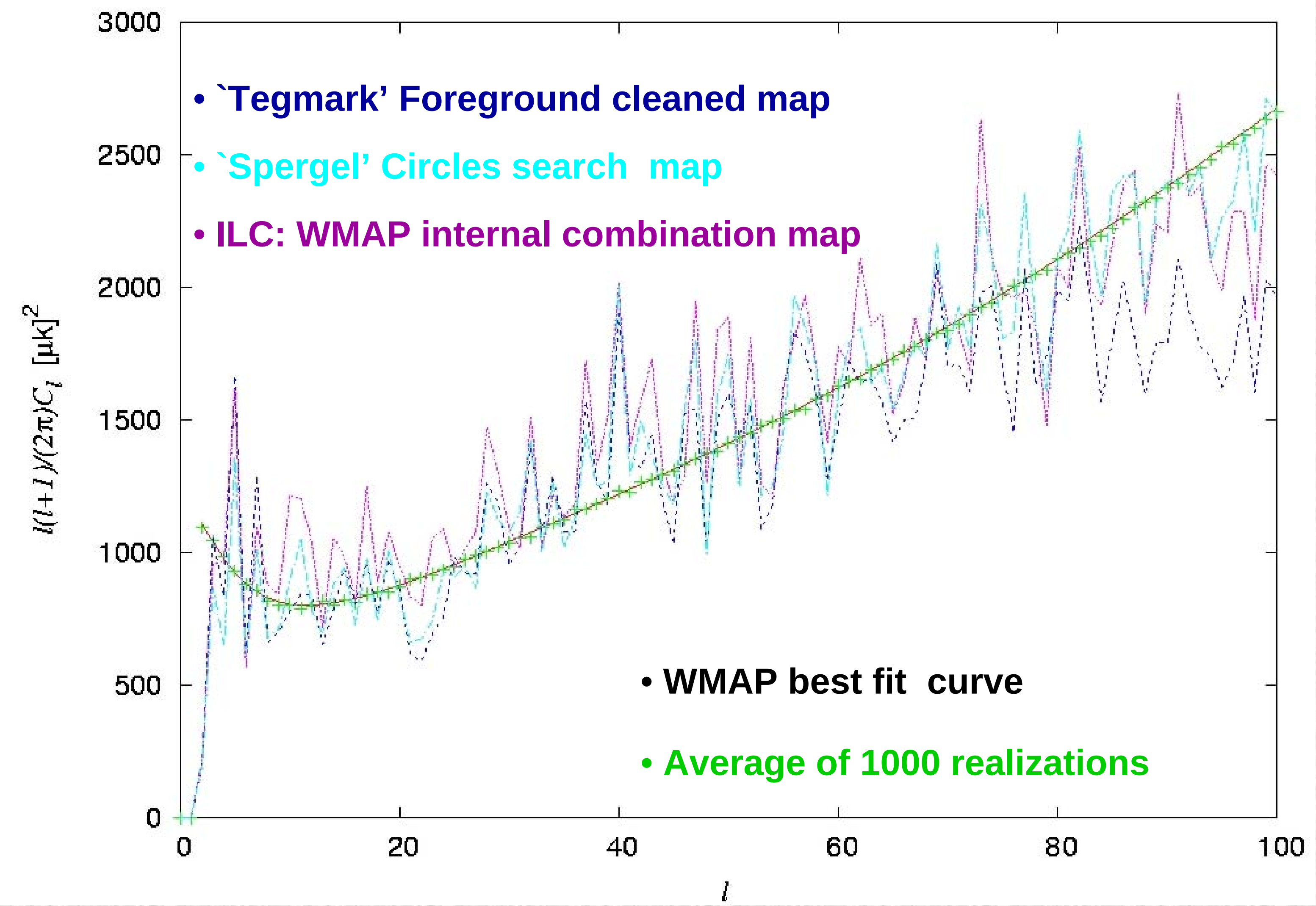
NASA/WMAP science team



Circles search *(Cornish, Starkman, Spergel, Komatsu 2004)*

Angular power spectra of the maps

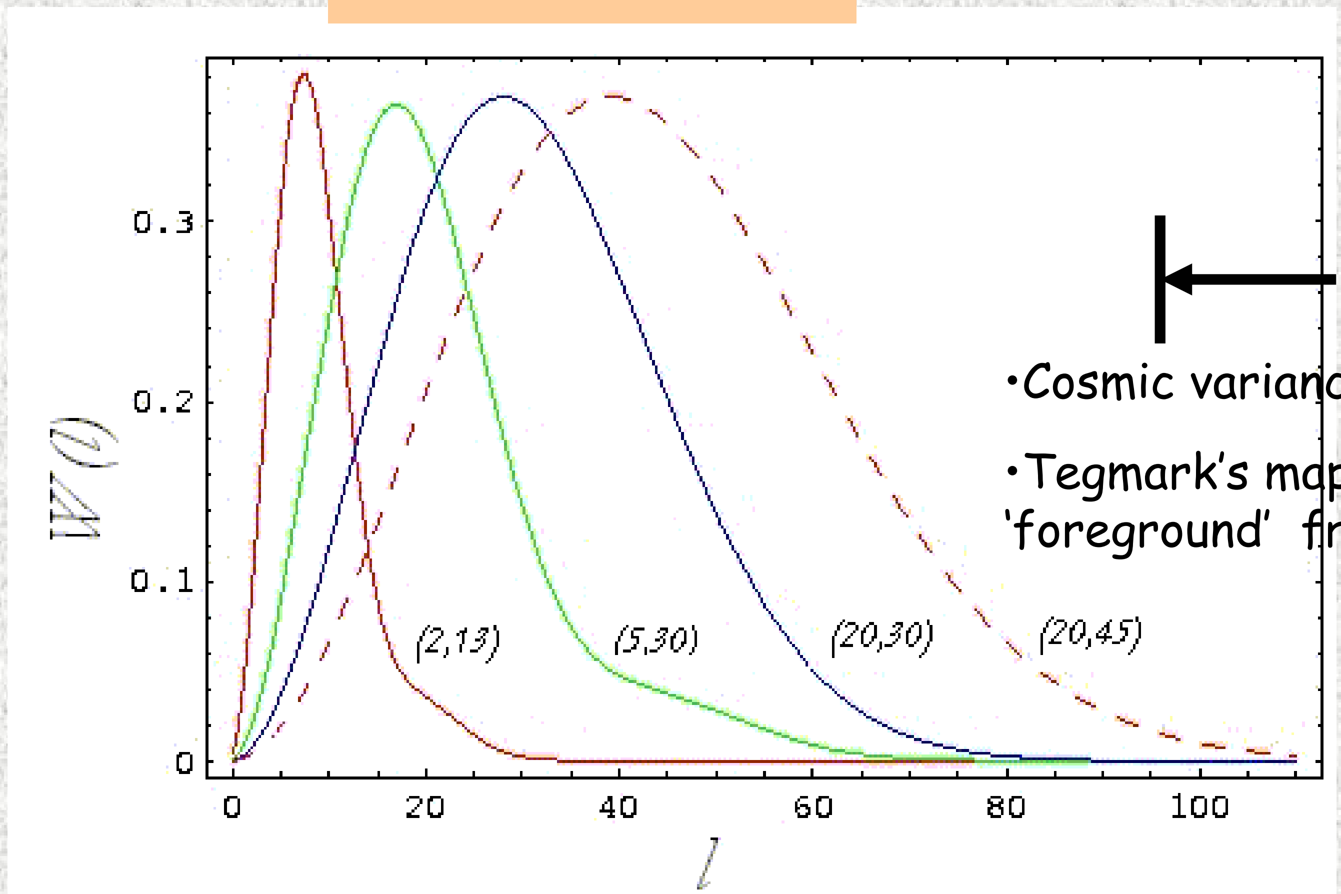
(compared to the WMAP best fit model)



Scanning the *l*-space with different windows

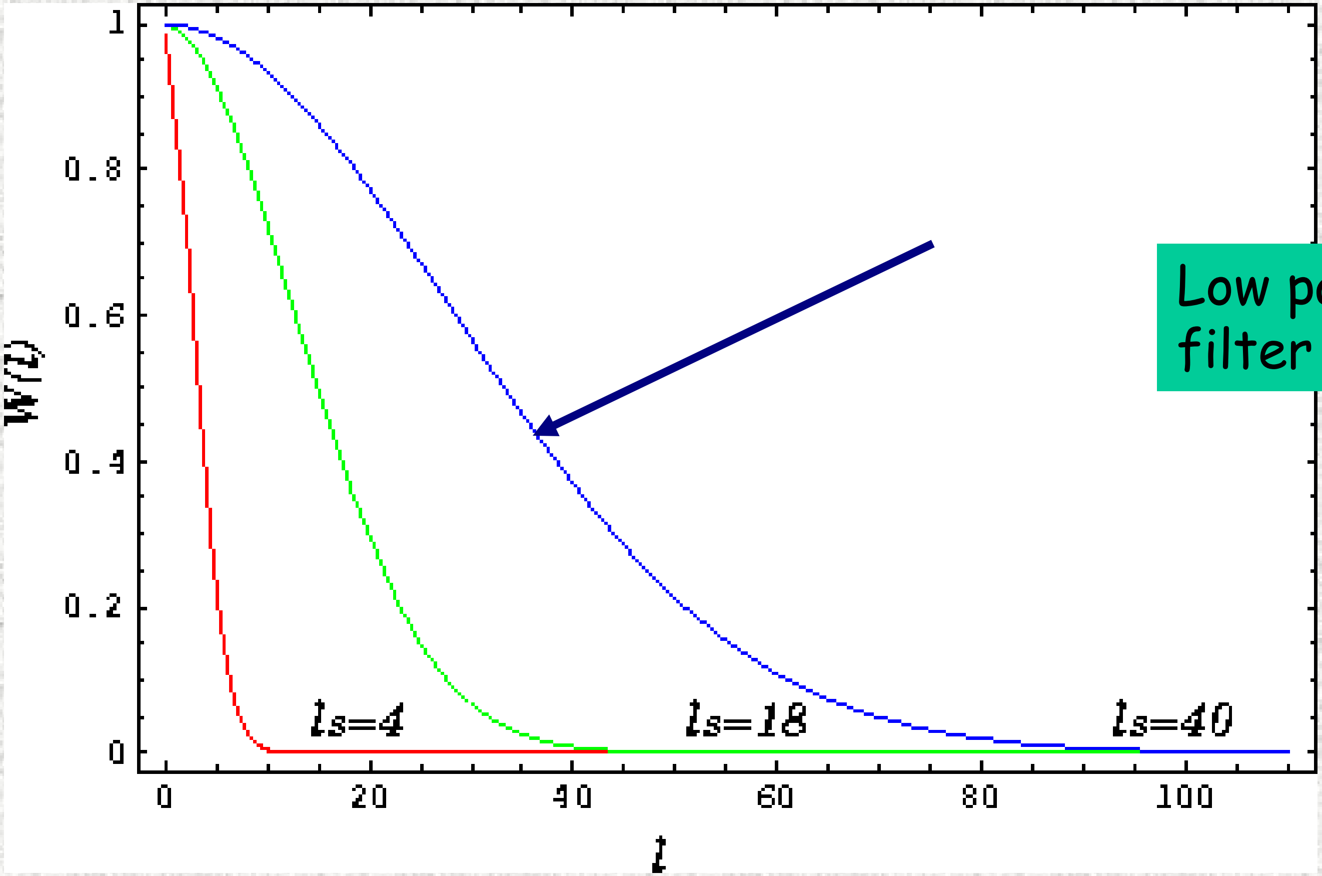
- *Maps can be filtered by isotropic window to retain power on certain angular scales, (eg., l~30 to 70)*

$$a_{lm} \rightarrow \sqrt{W_l} \quad a_{lm}$$



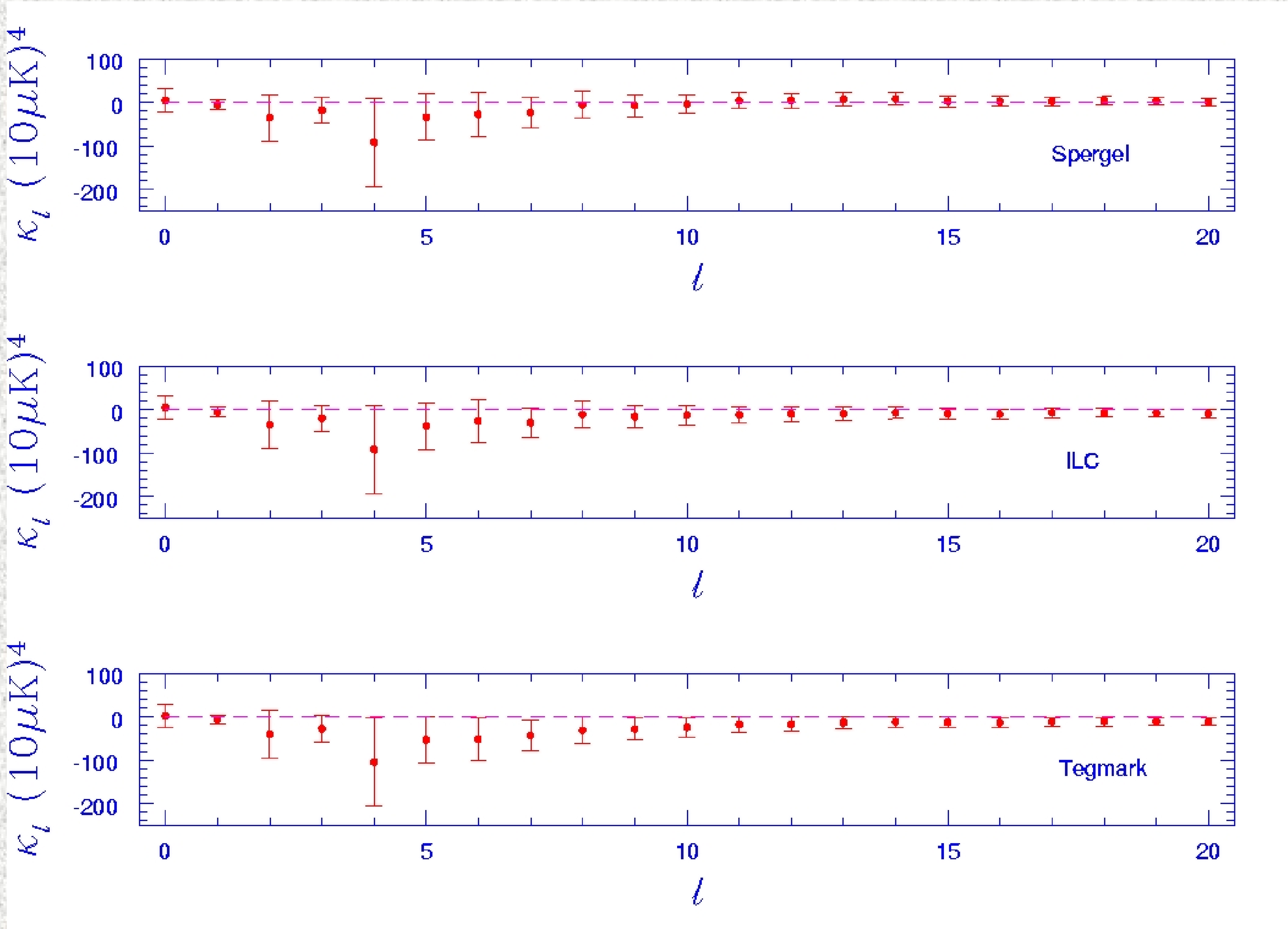
- Cosmic variance >> Noise
- Tegmark's map is 'foreground' free

Testing Statistical Isotropy of WMAP



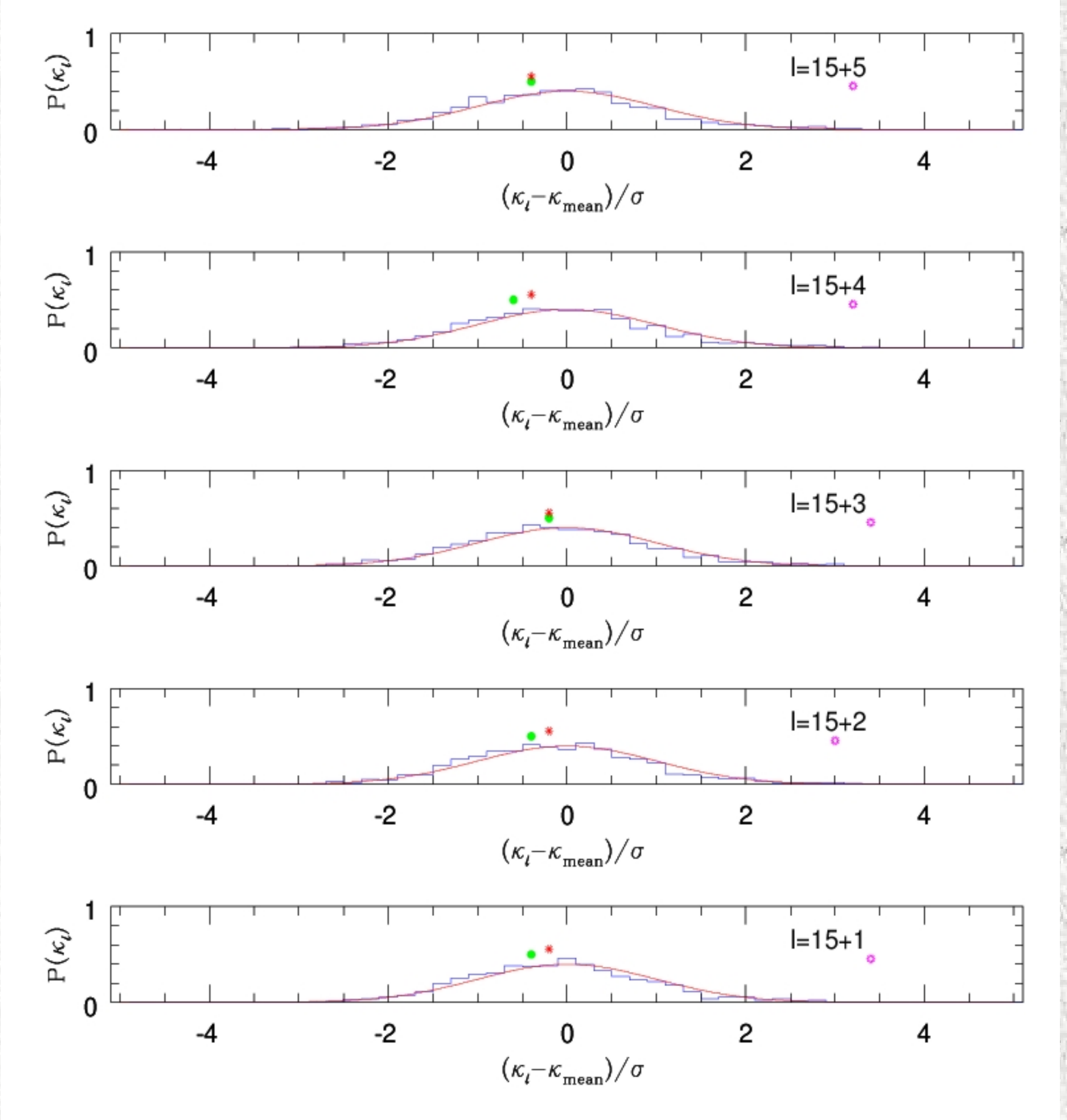
Low pass Gaussian filter at *l= 40*

(Hajian, TS, Cornish astro-ph/0406354)



(assuming WMAP best fit model)

Probability Distribution of BiPS



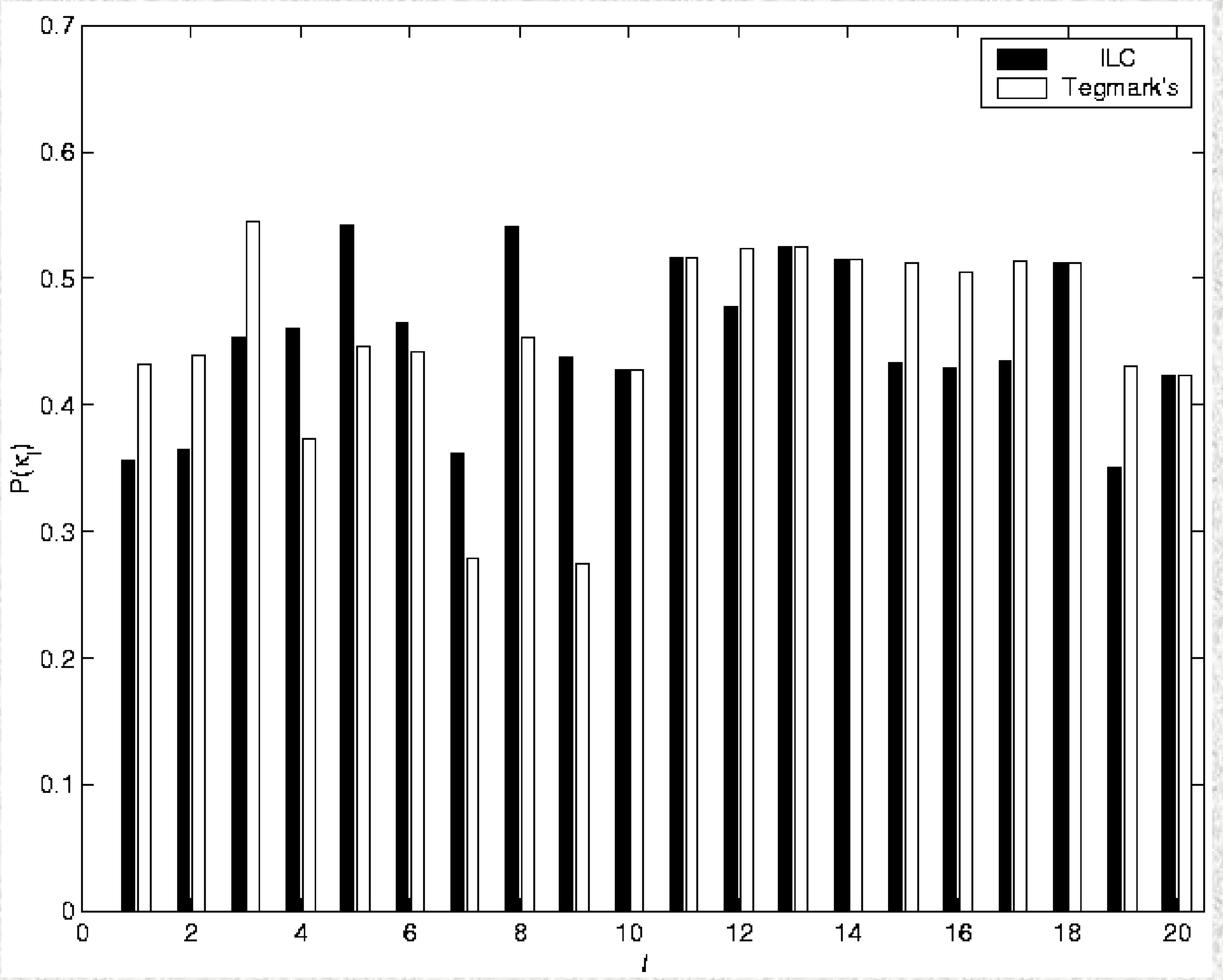
Obtained from measurements of 1000 simulated SI CMB maps.

Can compute a Bayesian probability of map being SI for each BiPS multipole

(Given theory Cl)

Probability of a Map being SI

(Hajian, TS, Cornish astro-ph/0406354)

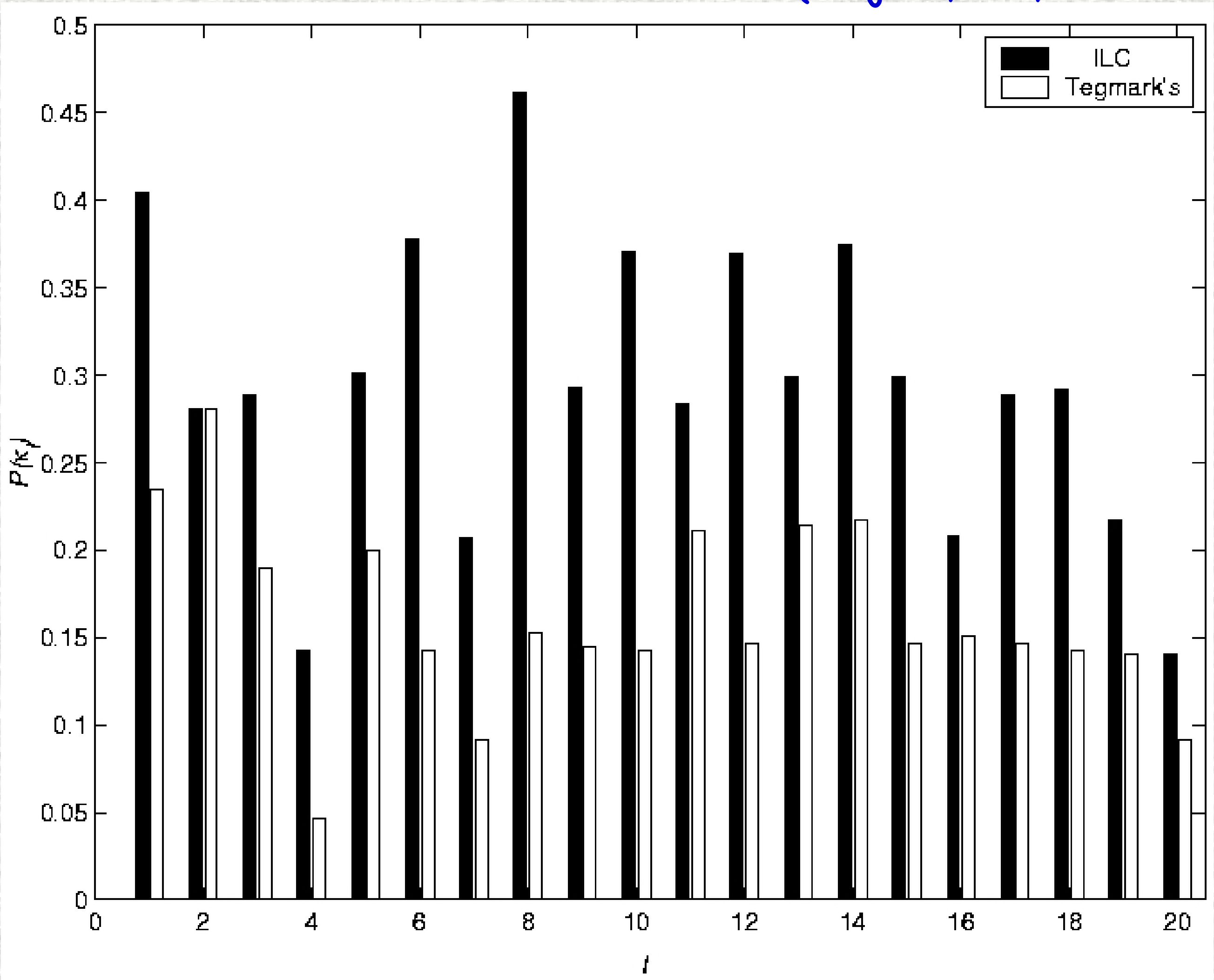


Bayesian probability

Band pass filter between *multipoles 20-30*

Probability of a Map being SI

(Hajian, TS, Cornish astro-ph/0406354)



Bayesian probability

Low pass Gaussian filter at $l = 40$

What does the null BiPS meaurement of CMB maps imply

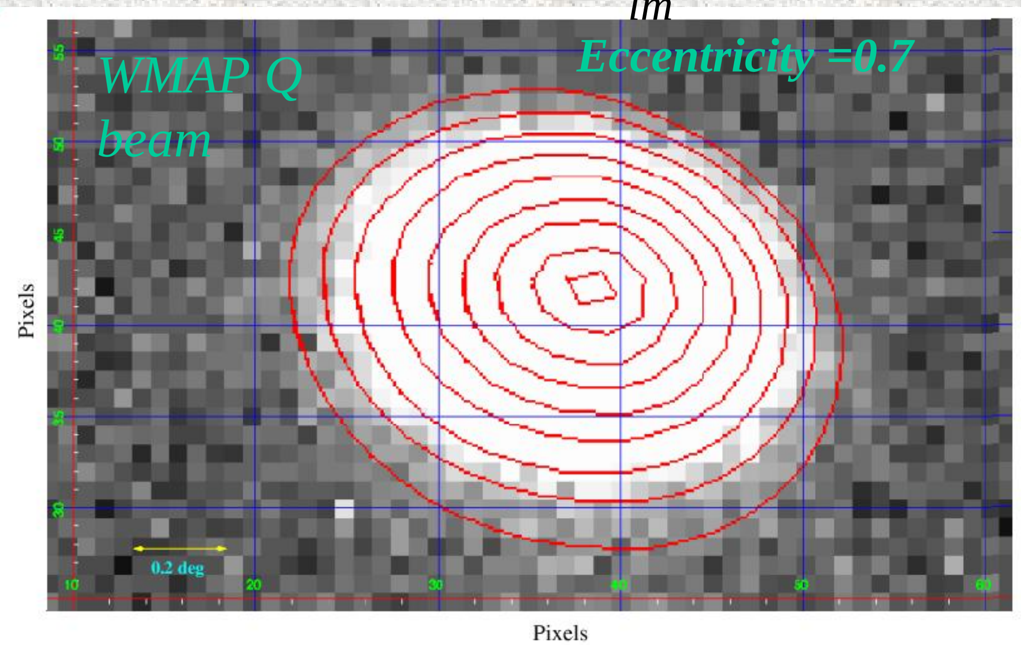
Sources of Statistical Anisotropy

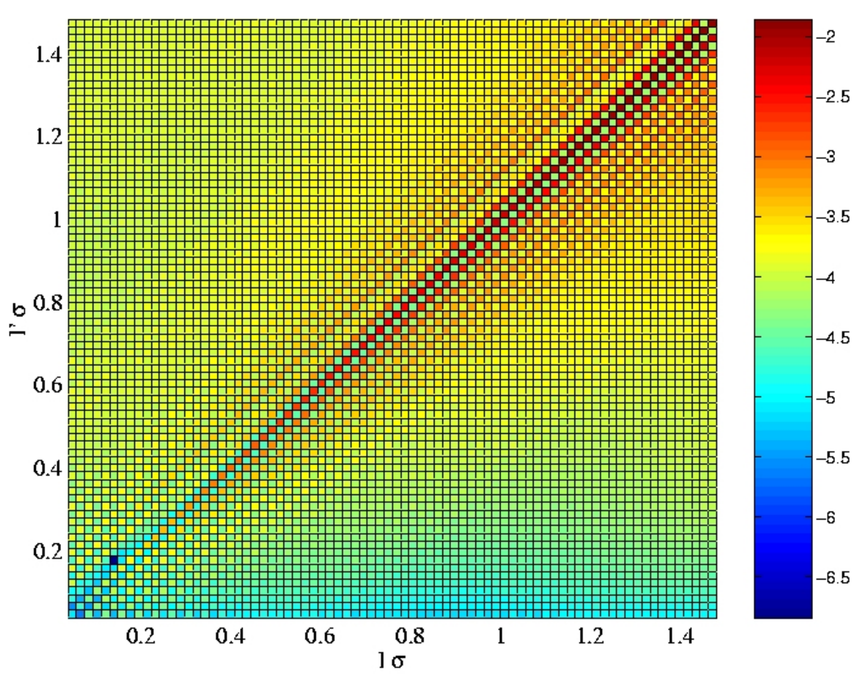
- Ultra large scale structure and cosmic topology.
- Primordial magnetic fields (based on Durrer et al. 98, Chen et al. 04).

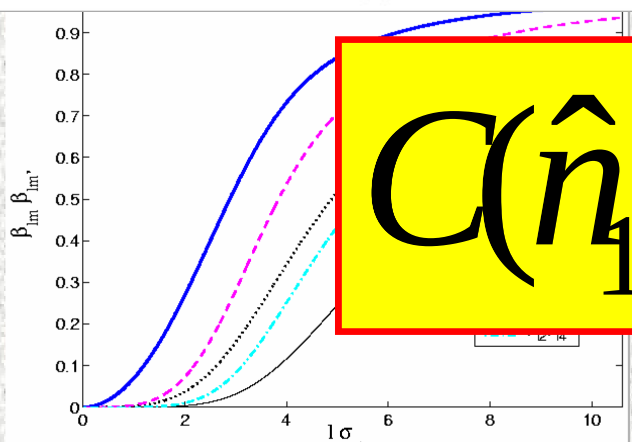
- Observational artifacts:
  - – Anisotropic noise
  - – Non-circular beam
  - – Incomplete/unequal sky coverage
  - – Residuals from foreground removal

Power spectrum estimation with non-circular beam

$$\text{Beam}: \; B(\hat{n},\hat{z}) = \sum_{lm} B_l \quad \beta_{lm}(\hat{n}) \quad Y_{lm}(\hat{n})$$





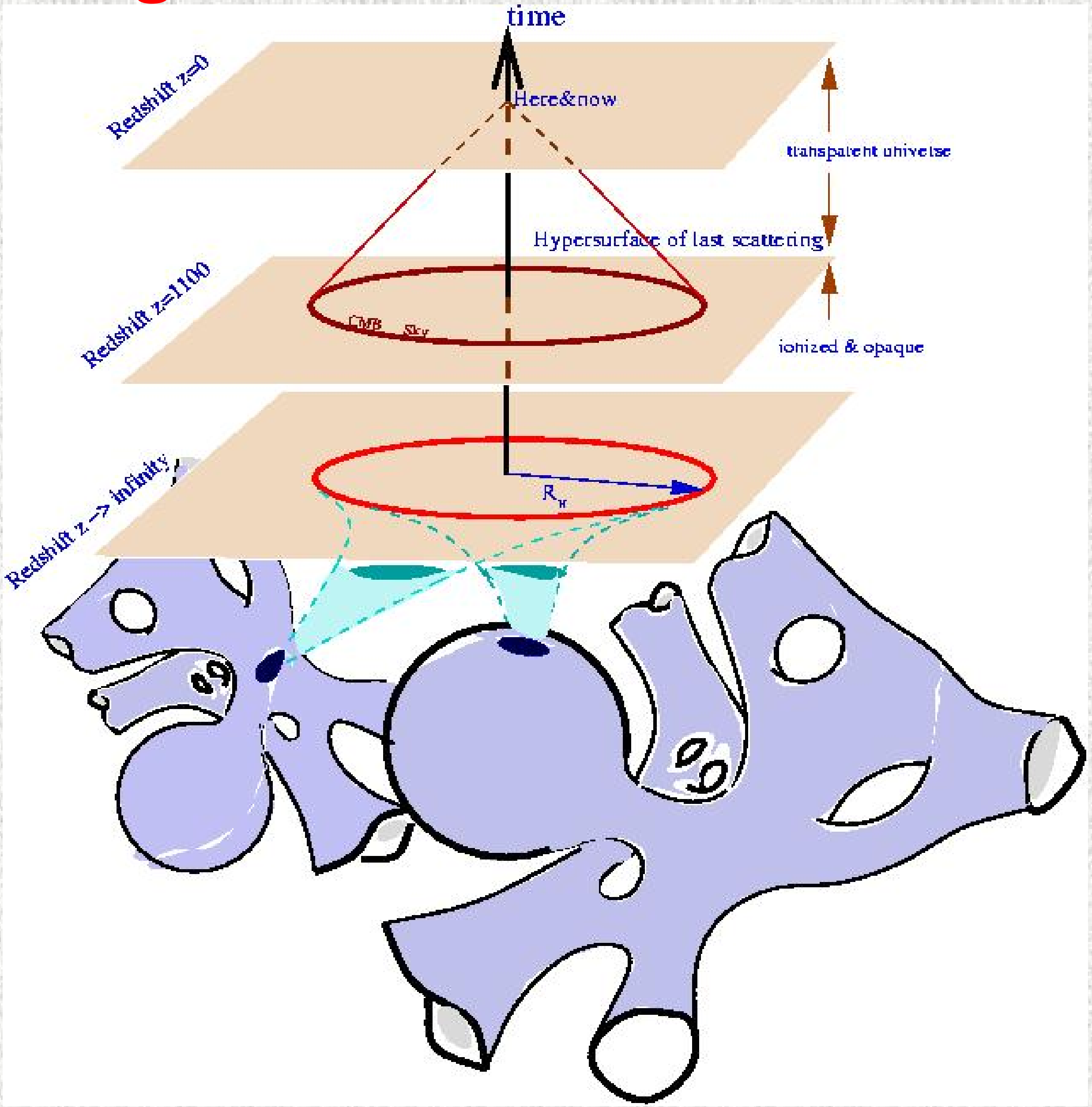


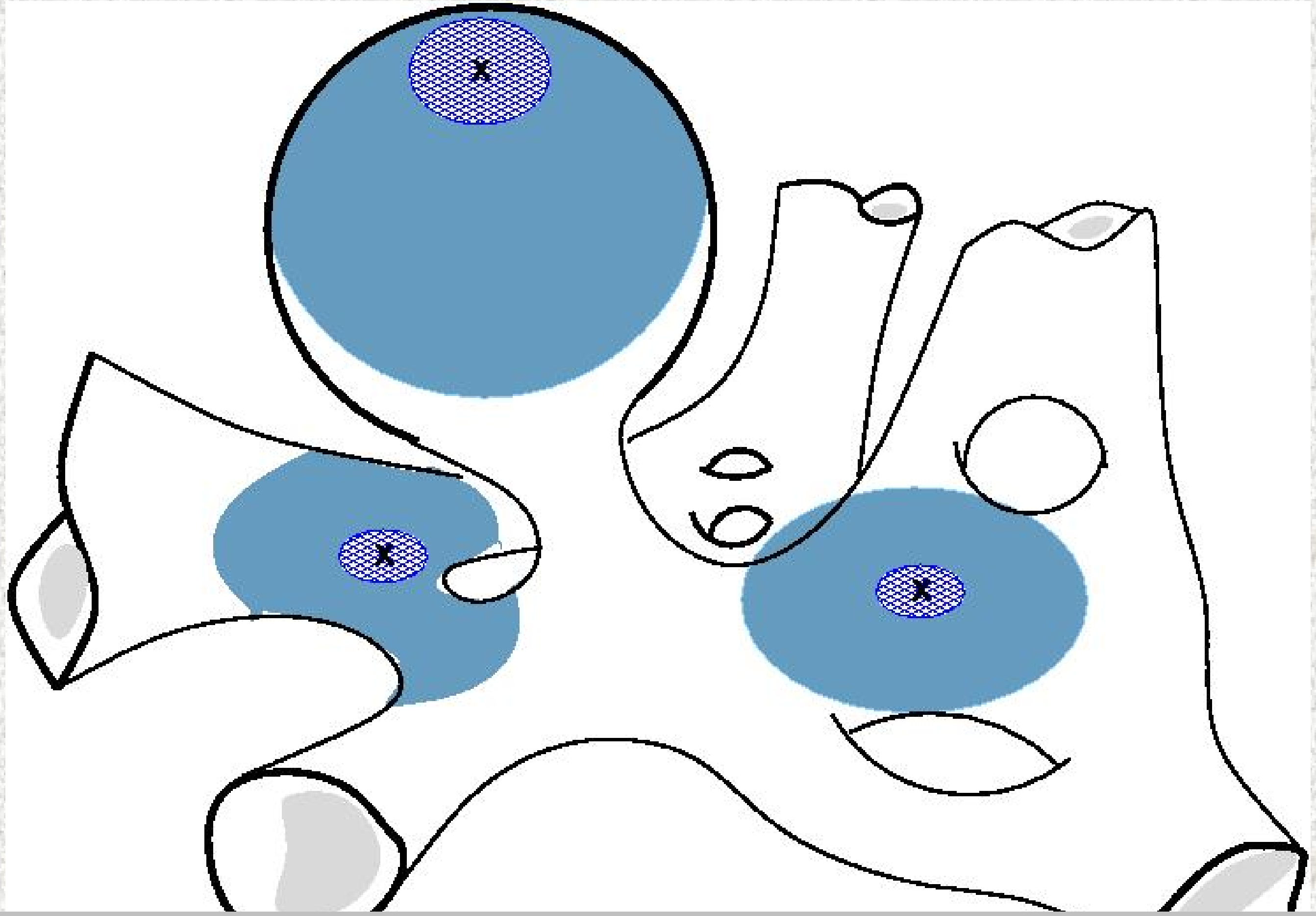
$$C(\hat{n}_1,\hat{n}_2) \neq C(\hat{n}_1 \bullet \hat{n}_2)$$

& effect on Cosmic variance

(S. Mitra, A. Sengupta, TS, 2003)

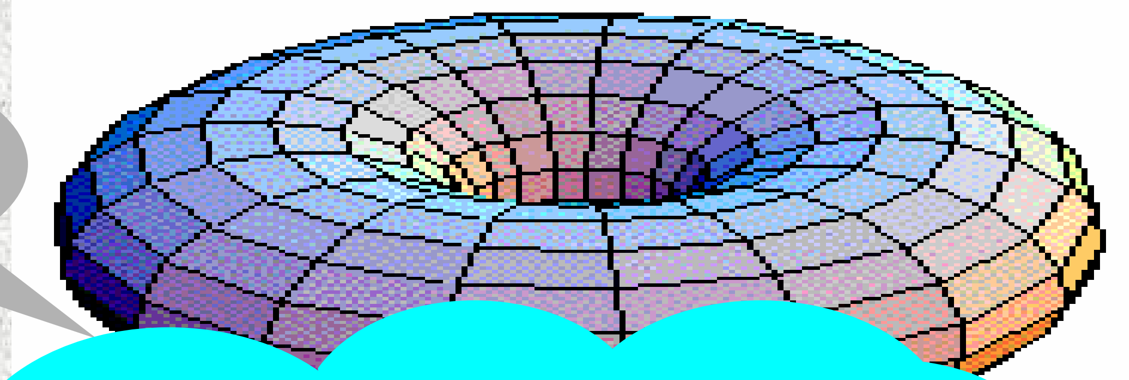
Ultra Large scale structure of the universe



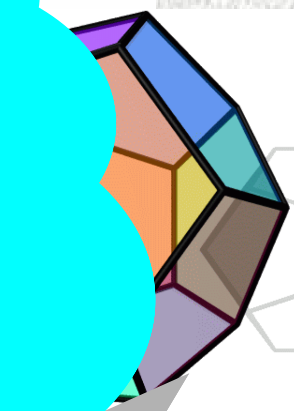


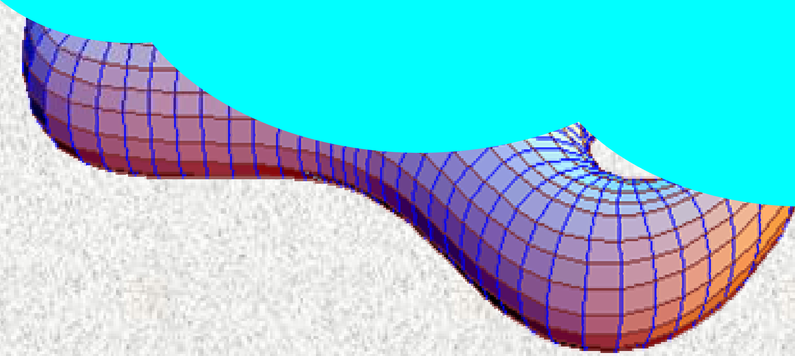
How Big is the Observable Universe ?

Relative to the local curvature & topological scales



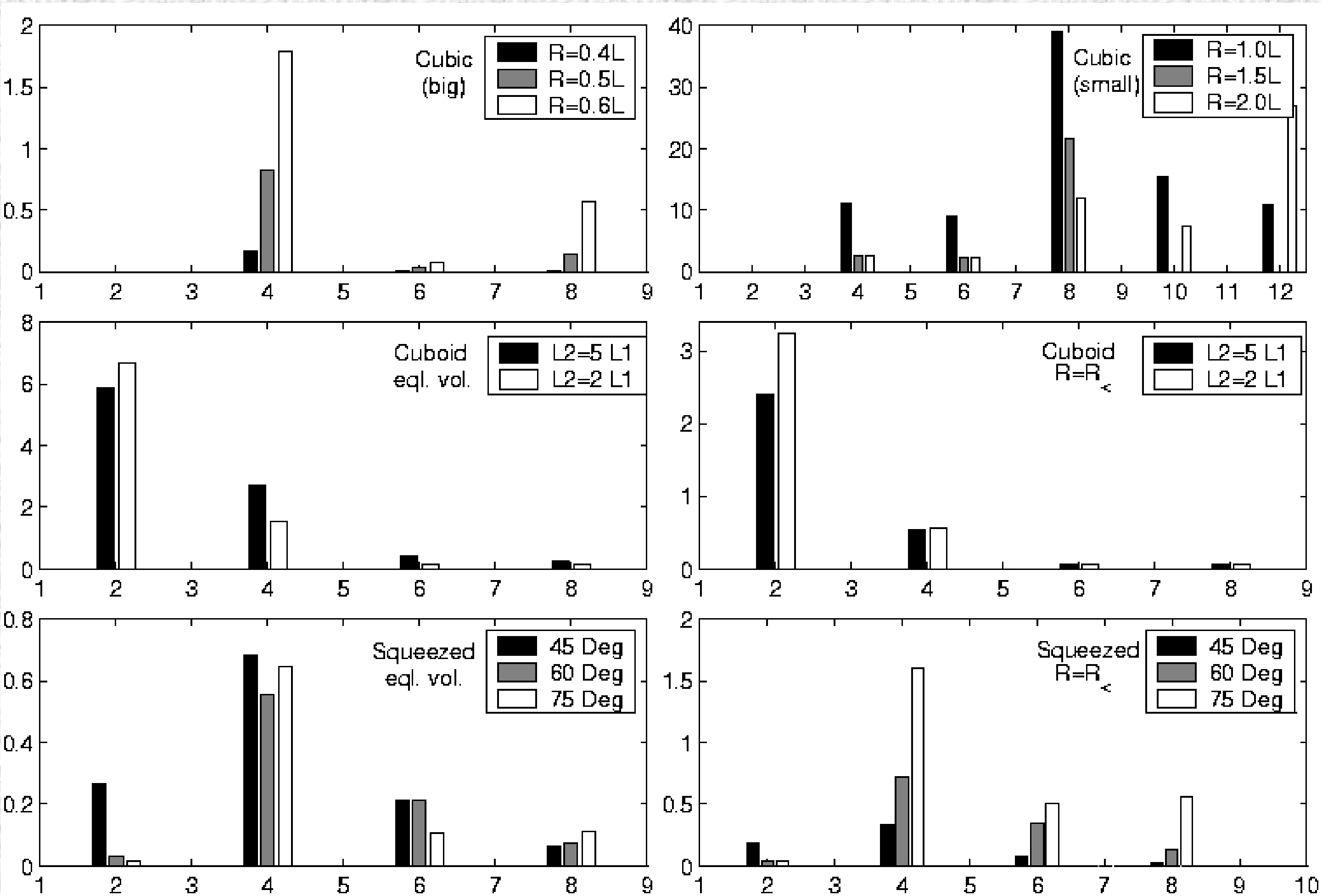
Cosmic Topology : Consider all Spaces of *Constant* Curvature

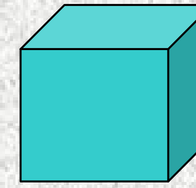


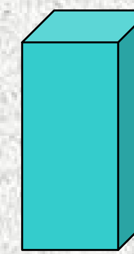


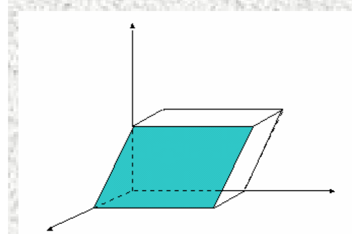
BiPS signature of Flat Torus spaces

κ_ℓ







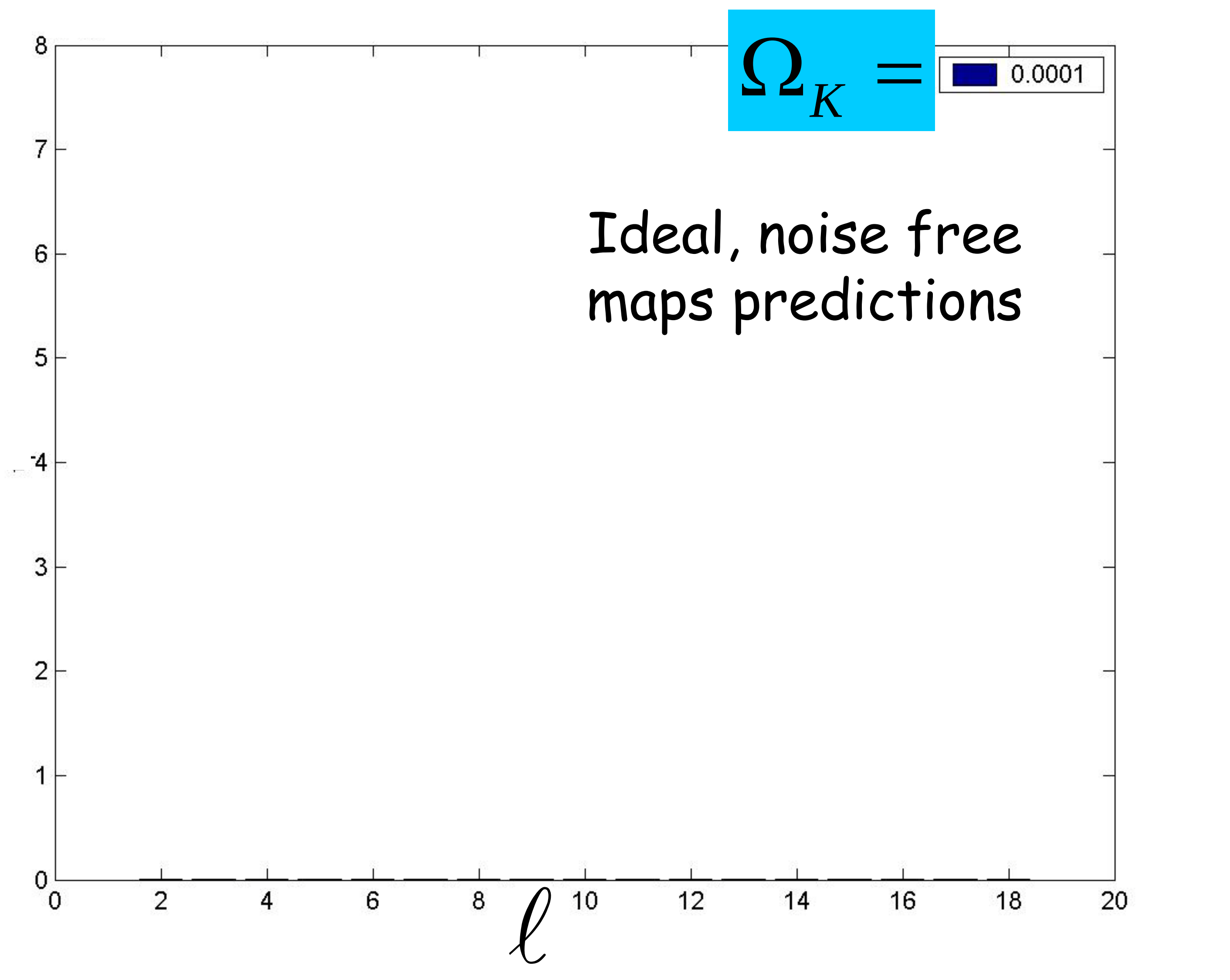


ℓ

Hajian & Souradeep (astro-ph/0301590)

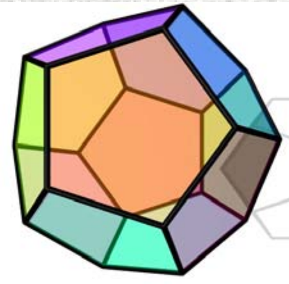
BiPS signature of a "soccer ball" universe

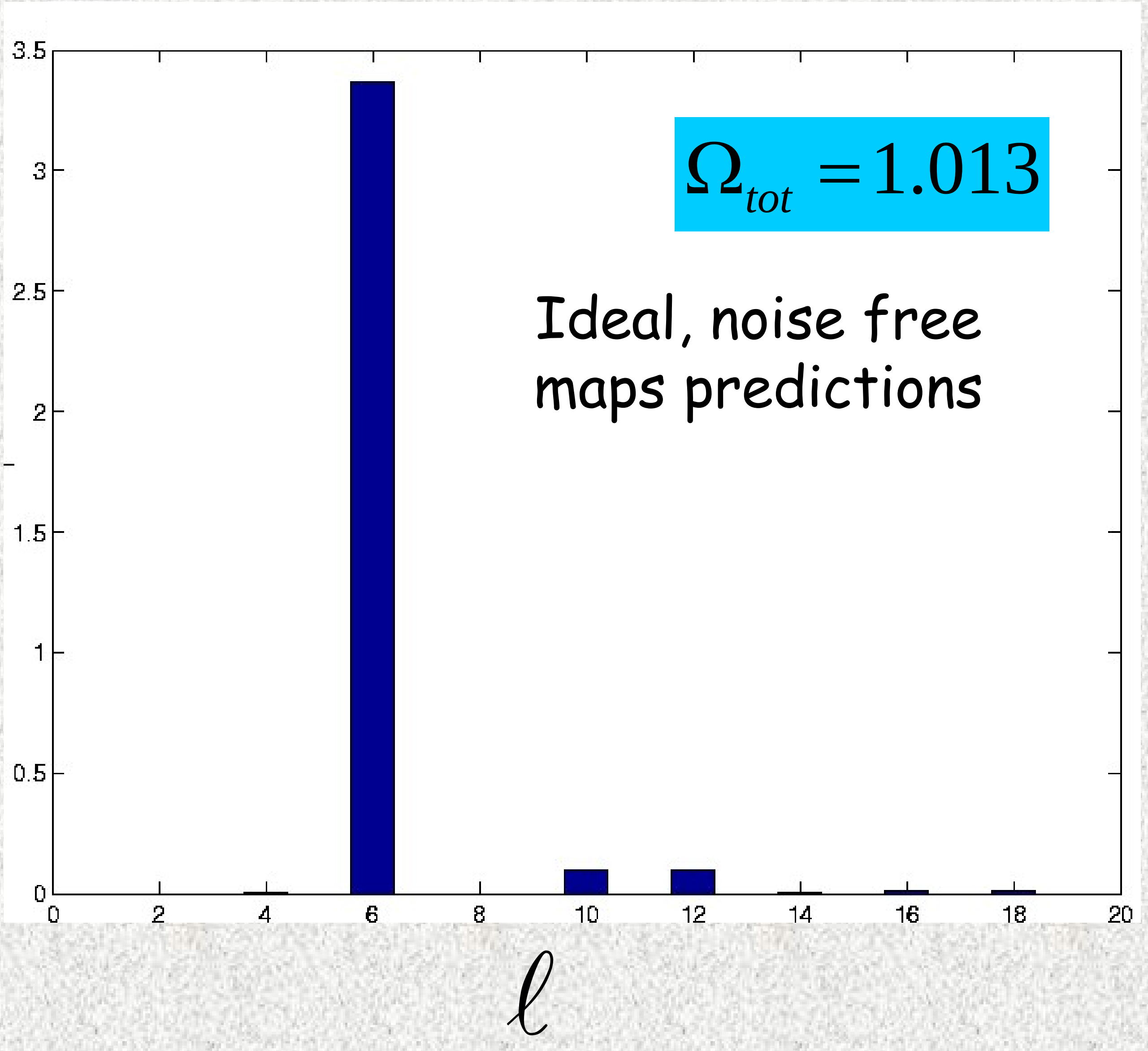
(Hajian, Pogosyan, TS, Contaldi, Bond : in progress.)



BiPS signature of a "soccer ball" universe

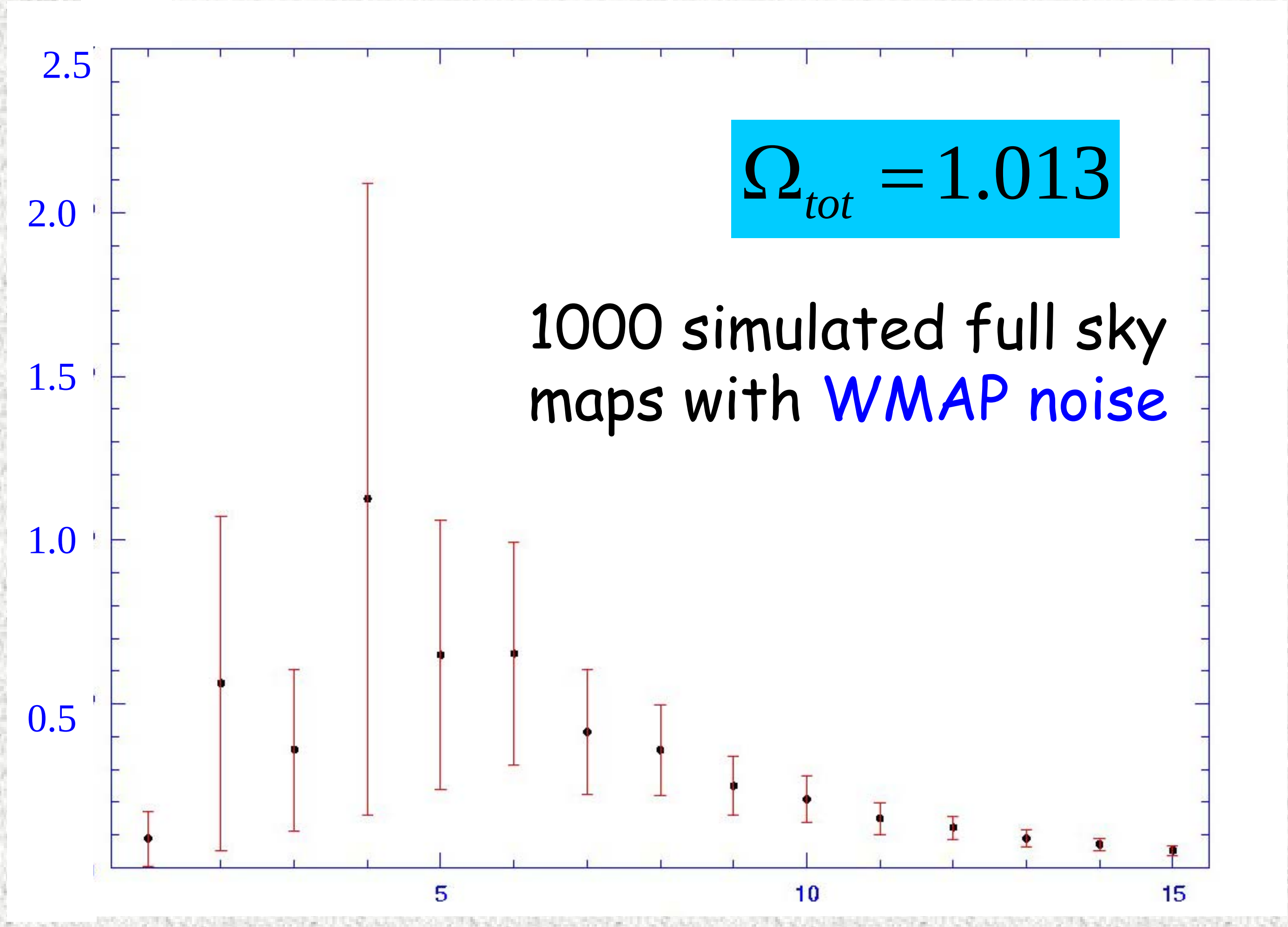
(Hajian, Pogosyan, TS, Contaldi, Bond : in progress.)





Measured BiPS for a "soccer ball" universe

(Hajian, Pogosyan, TS, Contaldi, Bond : in progress.)



Summary

- Propose BiPS as a generic measure for detecting and quantifying Statistical isotropy violations.

  ➔BiPS is **insensitive to the overall orientation** of SI breakdown (e.g., orientation of preferred axes). *Hence constraints are not orientation specific.*

  ➔ Computationally fast method

- Null results on some WMAP full sky maps.

  ➔ SI improves for a theory that predicts low power on low multipoles.

- Can constrain/detect cosmic topology and Ultra large scale structure, primordial magnetic fields..

  ➔BiPS promises to constrain Dodecahedron universe strongly.

- Diagnostic tool for observational artifacts.

Upcoming results, ongoing work & future plans

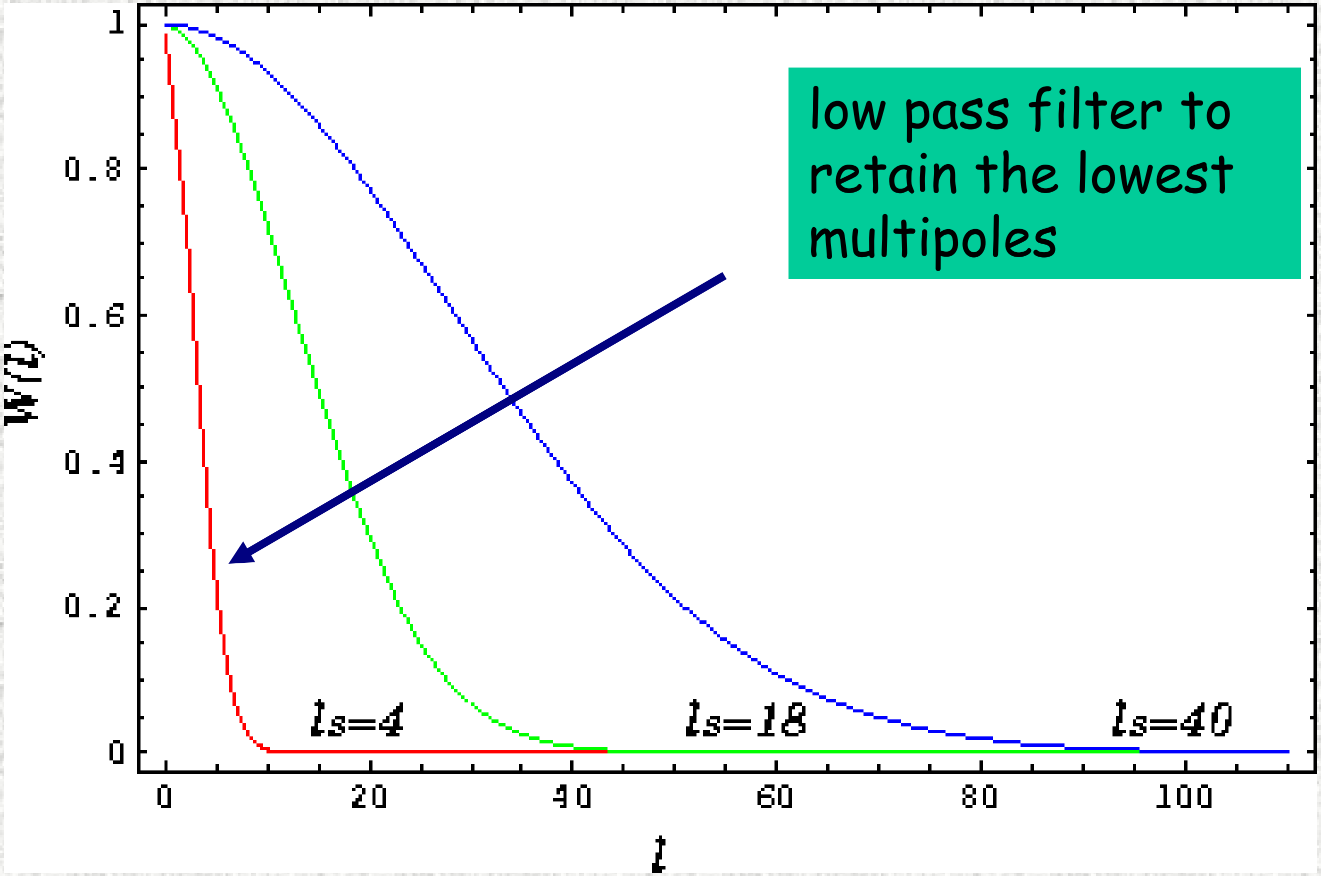
- Check for unusually large BiPoSH coefficients.
- WMAP results for a ‘frequentist’ BiPS measure.

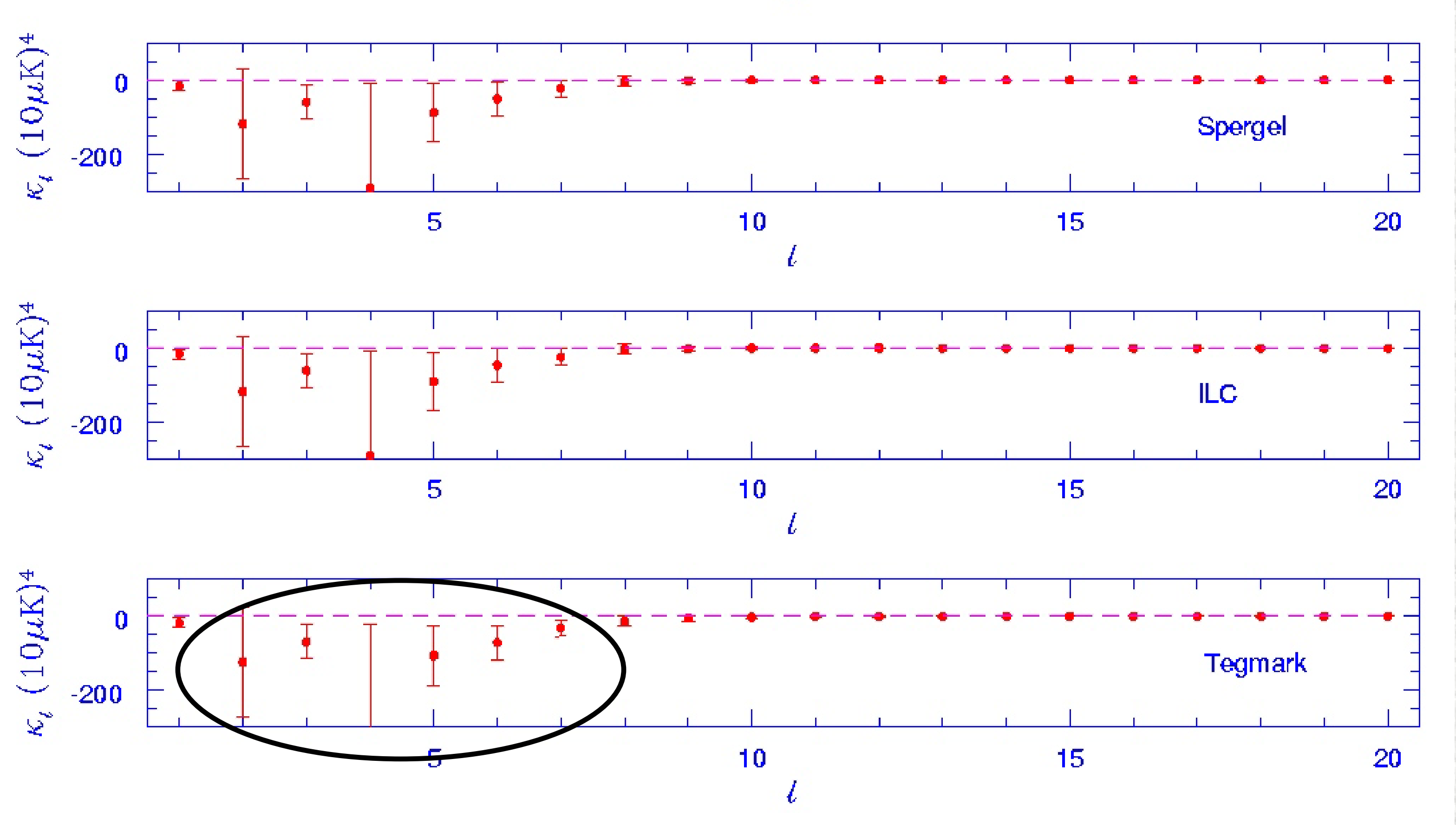
- BiPS constraints on cosmic topology.
- BiPS constraints on primordial magnetic fields
  (based on Chen et al. 04, Durrer et al. 02)

- BiPS interpretation of Eriksen et. al observation
- BiPS for residual foregrounds.
- **BipoSH & BiPS of CMB Polarization maps**
  (weak lensing shear fields ?)

- *Redoing cosmological parameter estimation using ‘optimal’ recovered spectrum.*

Testing Statistical Isotropy of WMAP



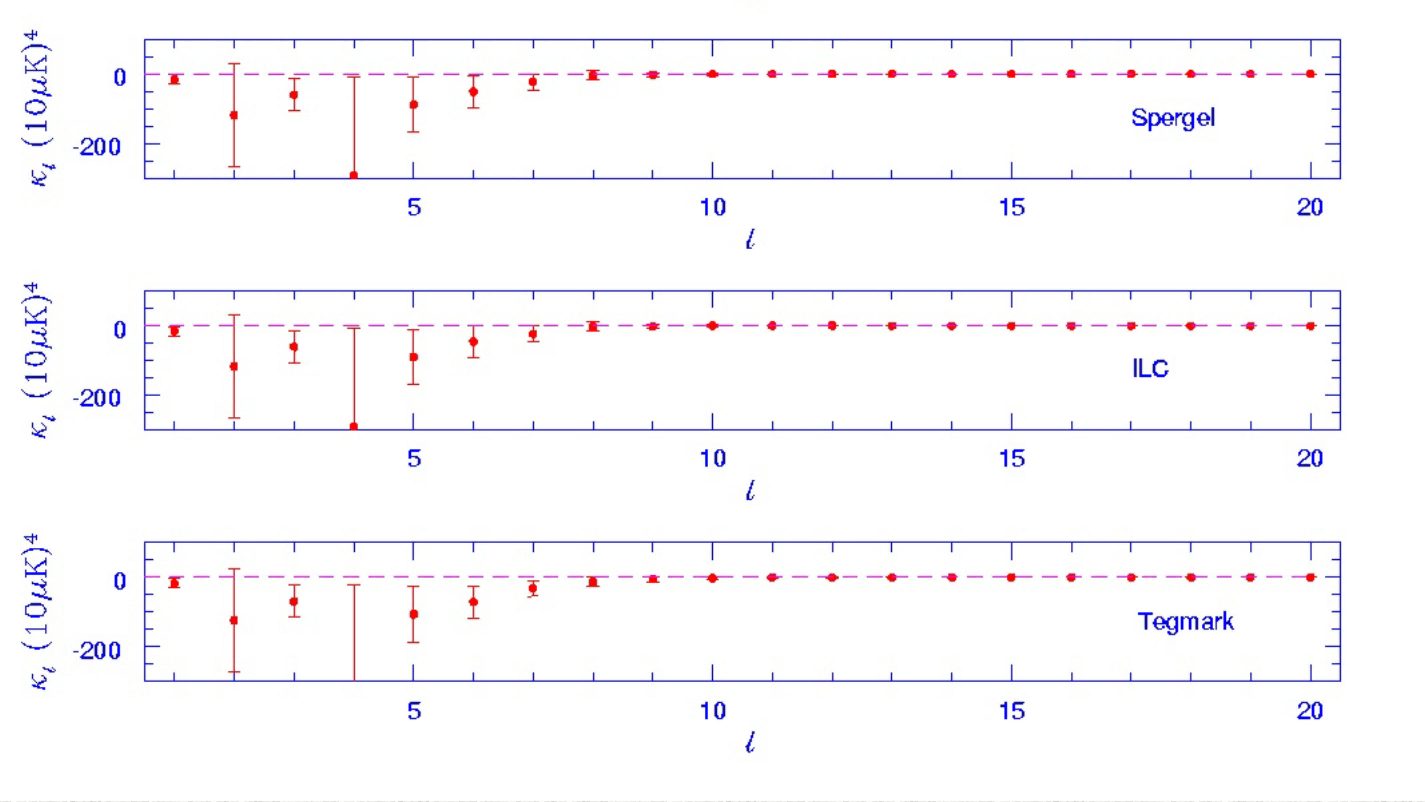


$\kappa_l\ (10\mu K)^4$
0
-200
5
10
15
20
l
Spergel
ILC
Tegmark

Statistical Isotropy of WMAP

Probability depend on the 'true' model

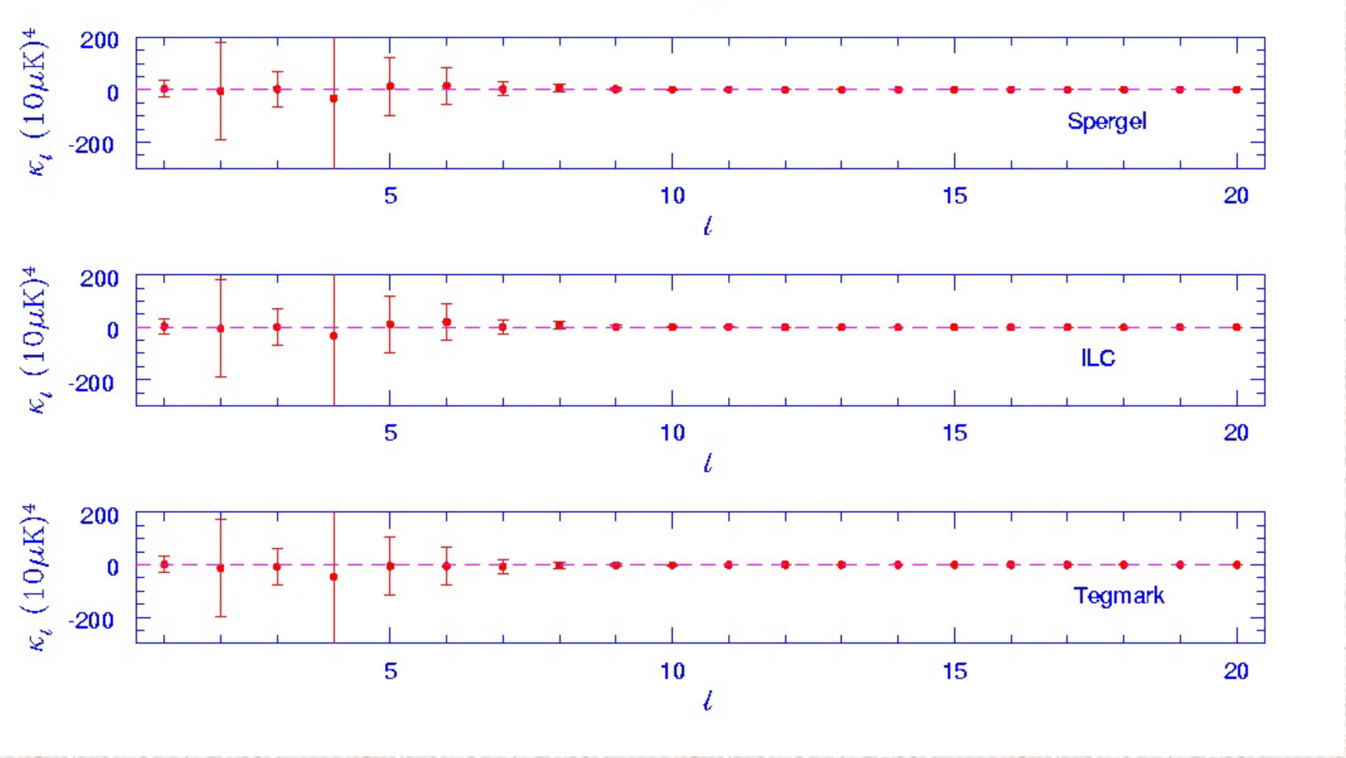
WMAP best fit theory spectrum over-predicts power on low multipoles



Statistical Isotropy of WMAP

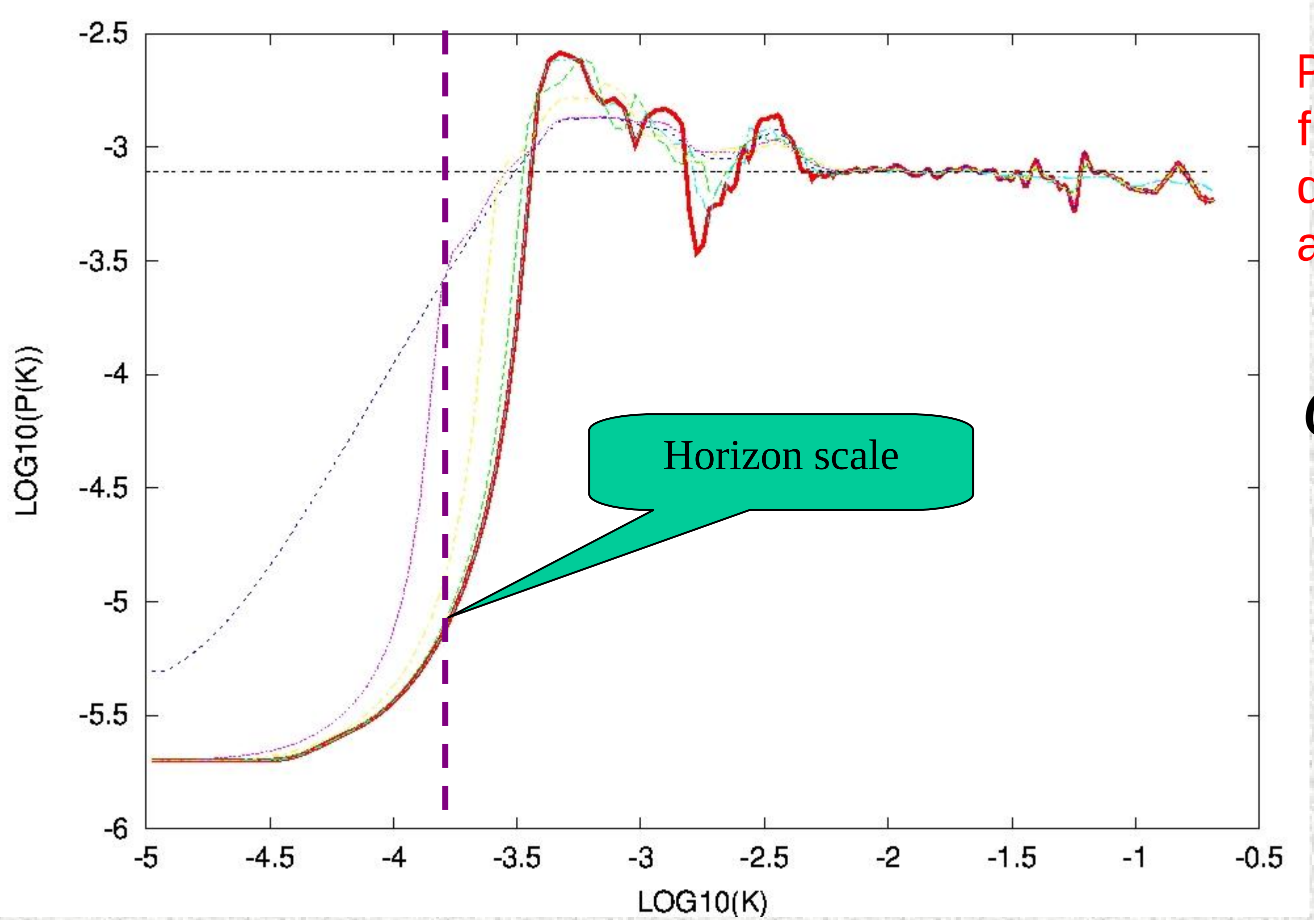
Probability depend on the 'true' model

WMAP maps are SI if the model fits the power on low multipoles !!!!



Recovering the primordial power spectrum

(Shafeiloo & Souradeep)



Primordial power spectrum from Early universe can deconvolved from CMB anisotropyspectrum

$$C_l = \int \frac{dk}{k} P(k) G_l(k)$$

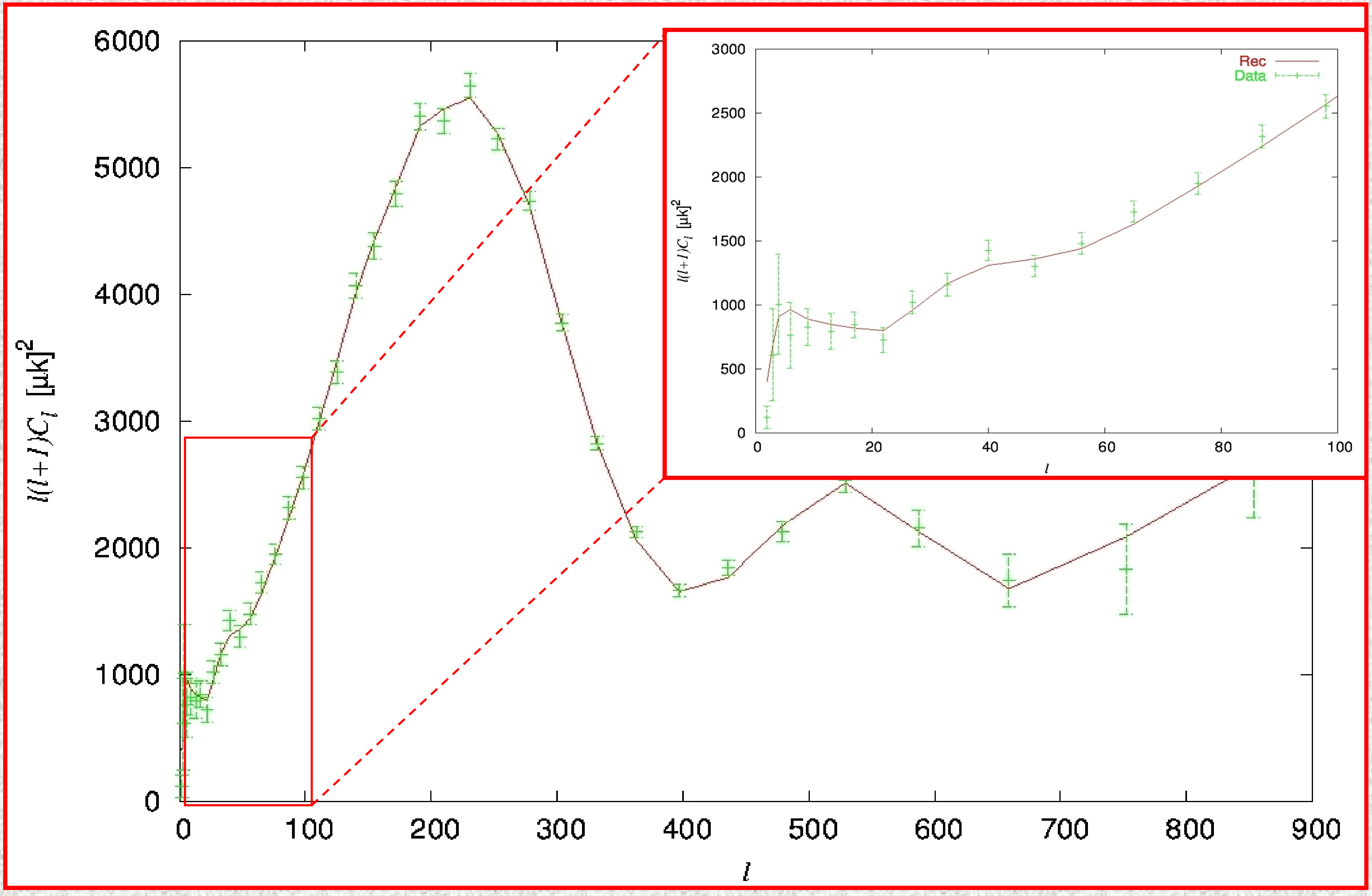
Improved Error sensitive iterative Richardson-Lucy deconvolution method

Recovered spectrum shows an infra-red cut-off on Horizon scale !!!

Is it cosmic topology ? Signature of pre-inflationary phase ? Trans-Planckian physics ? ….

Angular power spectrum from the recovered P(k)

(Shafieloo & Souradeep 2003)



Thank you !!!