Abstract

- Observed CMB polarization maps can be split into gradient-like (E) and curl-like (B) modes. I review the details of this decomposition, and the physical processes which give rise to the different types of polarization. The B-modes are a sensitive test of primordial gravitational waves as well as other things, and will need to be carefully distinguished from the larger E modes. I describe methods for performing E/B separation of the power spectra, as well as mode separation at the level of the map. I discuss the pros and cons of the various methods.

E/B Separation

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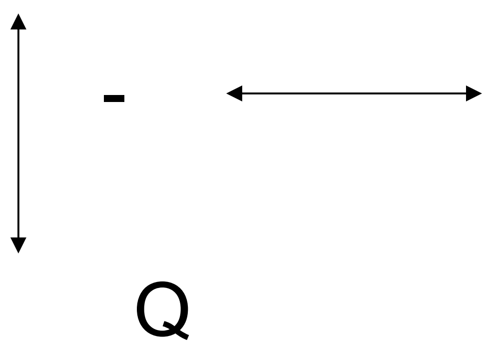
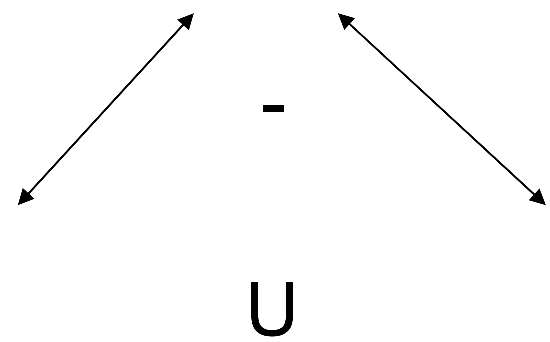
http://cosmologist.info

- Stokes parameters and E/B
- Possible E/B signals
- E/B separation
  - power spectrum methods
  - map-level methods

Polarization: Stokes' Parameters



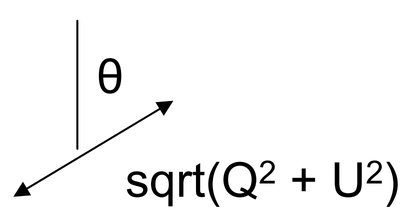


$Q \rightarrow -Q$, $U \rightarrow -U$ under 90 degree rotation

$Q \rightarrow U$, $U \rightarrow -Q$ under 45 degree rotation

Spin-2 field $Q + iU$
or Rank 2 trace free symmetric tensor

$$P = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}$$



$\theta = \frac{1}{2} \tan^{-1} U/Q$

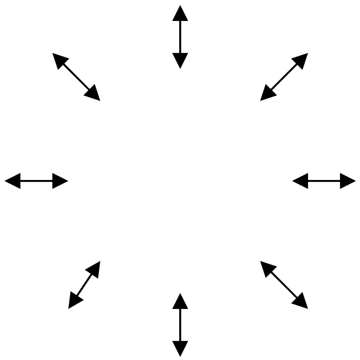
E and B

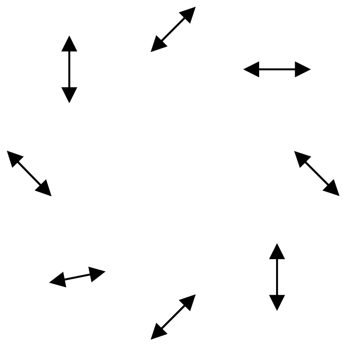
$$\mathcal{P}_{ab} = \nabla_{\langle a}\nabla_{b\rangle} P_E - \epsilon^{c}{}_{(a}\nabla_{b)}\nabla_c P_B$$

Trace free gradient:
E polarization

Curl:
B polarization

e.g.





2D vector analogy: $V_i = \nabla_i \Phi + \epsilon_i{}^{j}\nabla_j \chi,$

E and B harmonics

- Expand scalar P_E and P_B in spherical harmonics
- Expand P_{ab} in tensor spherical harmonics

$$\mathcal{P}_{ab} = \frac{1}{\sqrt{2}} \sum_{lm} \left(E_{lm} \, Y^{G}_{(lm)ab} + B_{lm} \, Y^{C}_{(lm)ab} \right)$$

$$E_{lm} = \sqrt{2} \int_{4\pi} \mathrm{d}S \, Y^{G\,ab*}_{(lm)} \mathcal{P}_{ab} \qquad B_{lm} = \sqrt{2} \int_{4\pi} \mathrm{d}S \, Y^{C\,ab*}_{(lm)} \mathcal{P}_{ab}$$

Harmonics are orthogonal over the full sky:

<u>E/B decomposition is exact and lossless on the full sky</u>

Zaldarriaga, Seljak: astro-ph/9609170
Kamionkowski, Kosowsky, Stebbins: astro-ph/9611125

CMB Signals

- E polarization from scalar, vector and tensor modes
- B polarization only from vector and tensor modes (curl grad = 0) + non-linear scalars

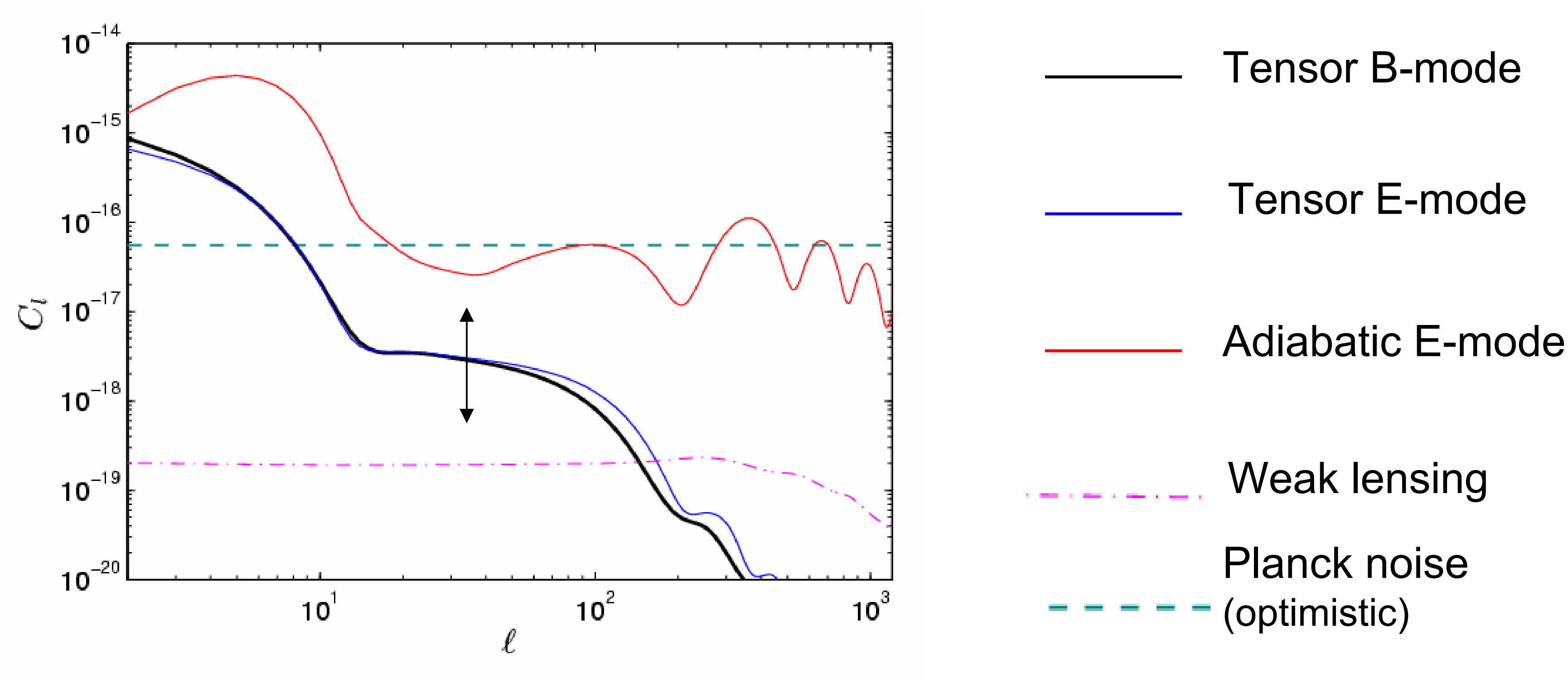
Average over possible realizations (statistically isotropic):

$$\langle E^*_{l'm'} E_{lm} \rangle = \delta_{l'l} \delta_{m'm} C_l^{EE} \qquad \langle B^*_{l'm'} B_{lm} \rangle = \delta_{l'l} \delta_{m'm} C_l^{BB}$$

Parity symmetric ensemble: $\langle E^*_{l'm'} B_{lm} \rangle = 0$

Power spectra contain all the useful information if the field is Gaussian

CMB polarization from primordial gravitational waves (tensors)

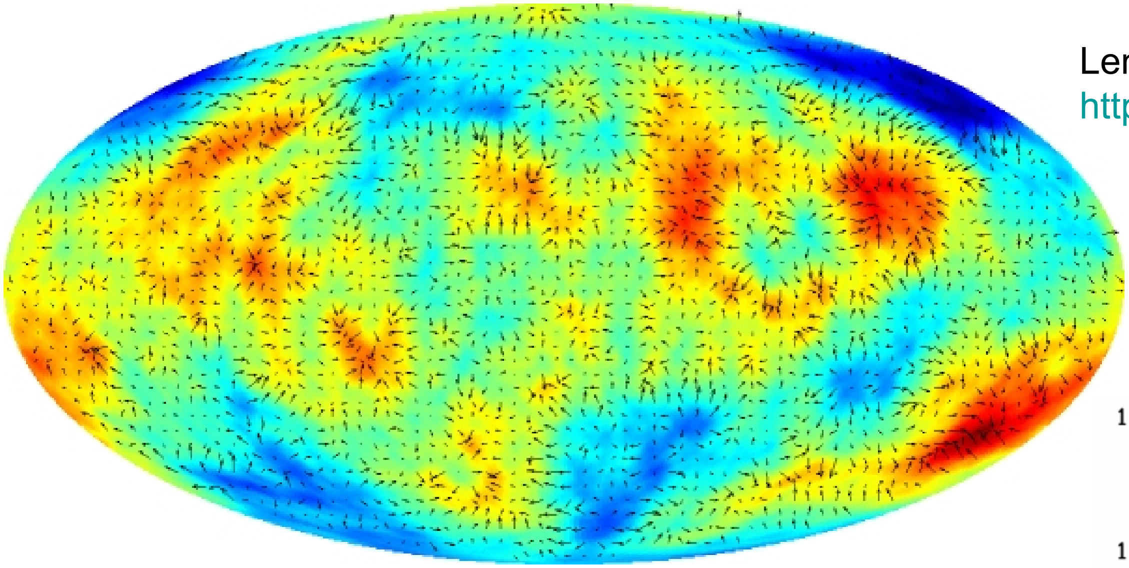


- Amplitude of tensors unknown
- Clear signal from B modes – there are none from scalar modes
- Tensor B is always small compared to adiabatic E

Seljak, Zaldarriaga: astro-ph/9609169

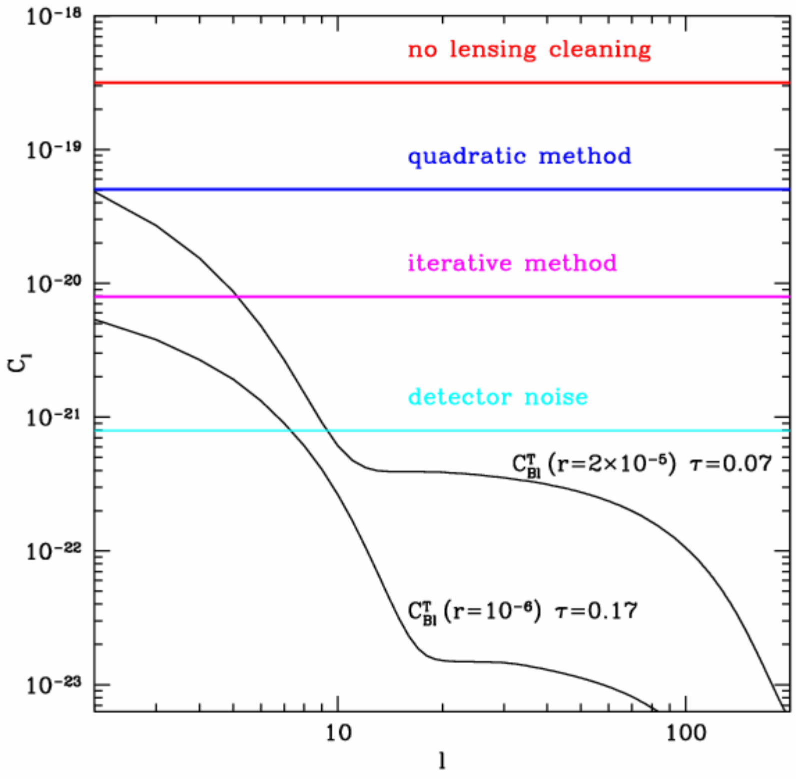
• Weak lensing of scalar mode E polarization

Zaldarriaga, Seljak: astro-ph/9803150; Hu: astro-ph/0001303



LensPix sky simulation code:
http://cosmologist.info/lenspix

Seljak, Hirata: astro-ph/0310163



Can be largely subtracted if only primordial tensors + lensing

Hirata, Seljak : astro-ph/0306354
Okamoto, Hu: astro-ph/0301031

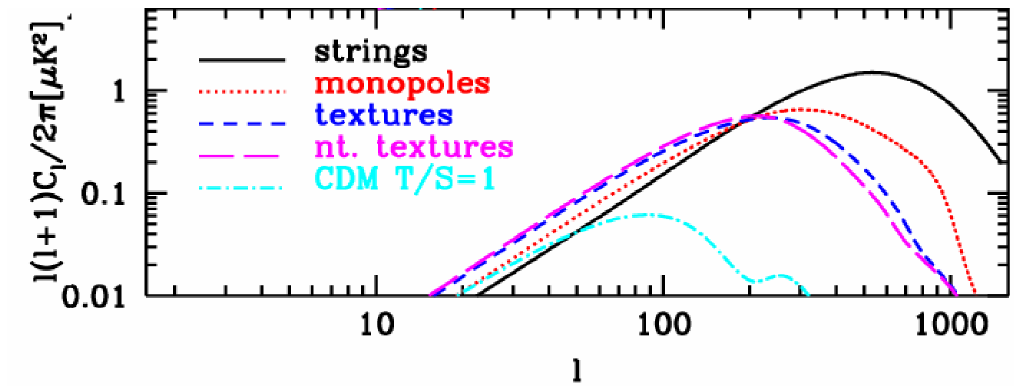
Other B-modes?

•Topological defects

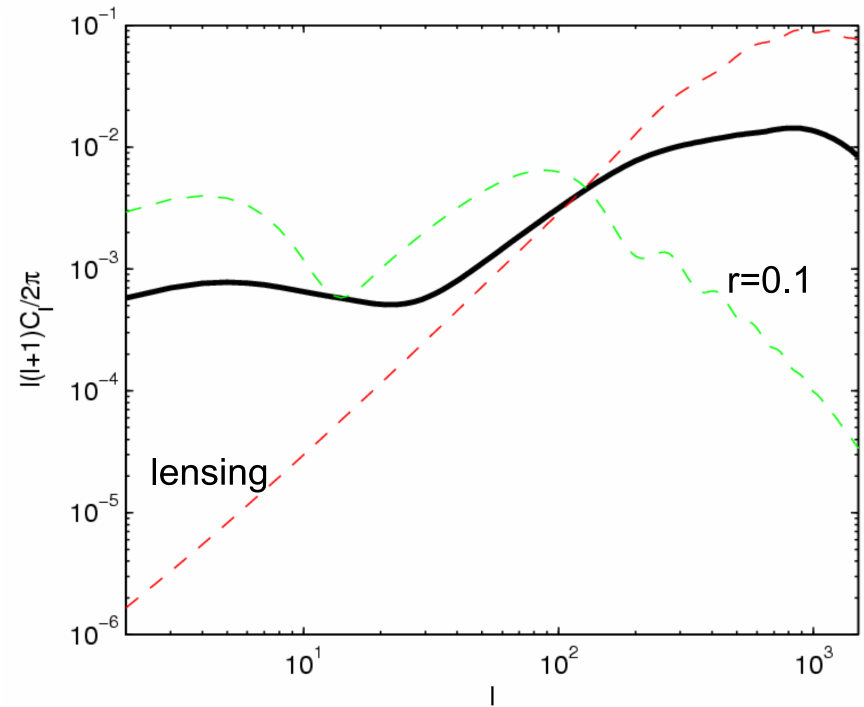
Seljak, Pen, Turok: astro-ph/9704231

Non-Gaussian signals

global defects:



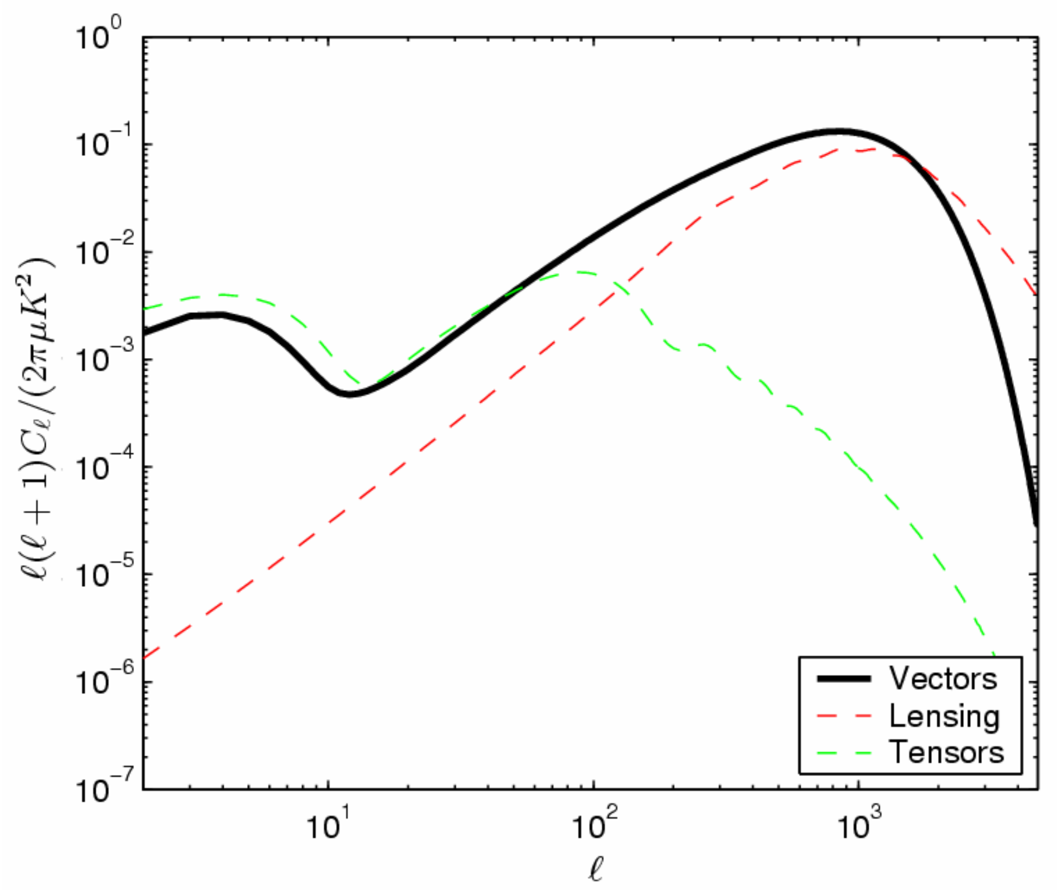
10% local strings from brane inflation:



Pogosian, Tye, Wasserman, Wyman: hep-th/0304188

• Regular vector mode: 'neutrino vorticity mode'

- logical possibility but unmotivated (contrived). Spectrum unknown.

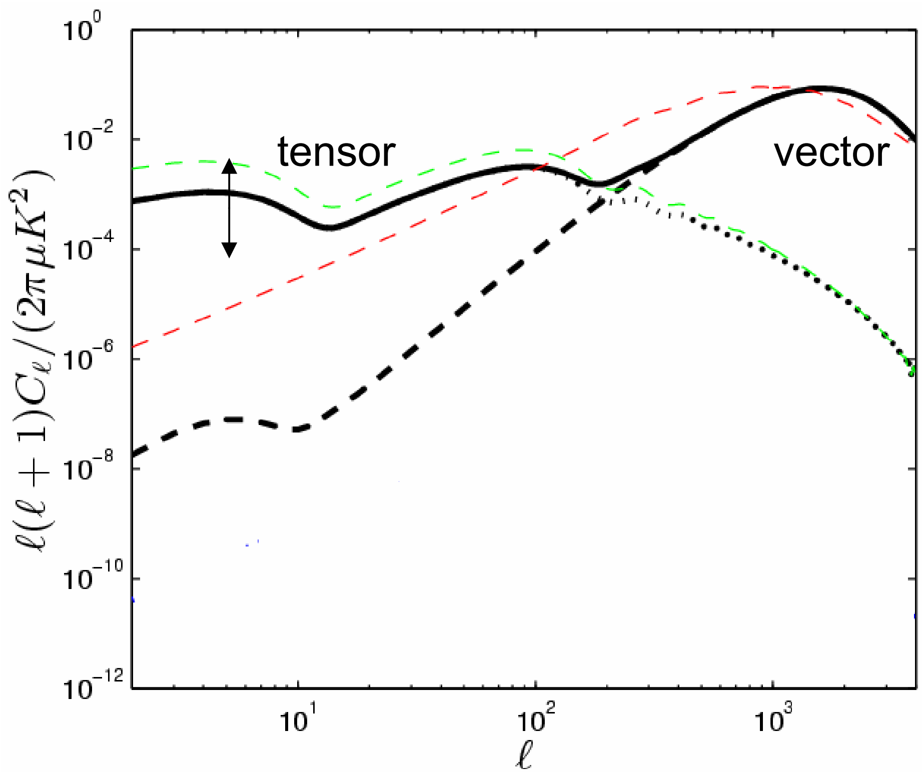


Similar to gravitational wave spectrum on large scales: distinctive small scale

Lewis: astro-ph/0403583

- Primordial magnetic fields
  - amplitude possibly right order of magnitude; not well motivated theoretically
  - contribution from sourced gravity waves (tensors) and vorticity (vectors)

e.g. Inhomogeneous field B = 3x10^{-9} G, spectral index n = -2.9



Tensor amplitude uncertain.

Non-Gaussian signal.

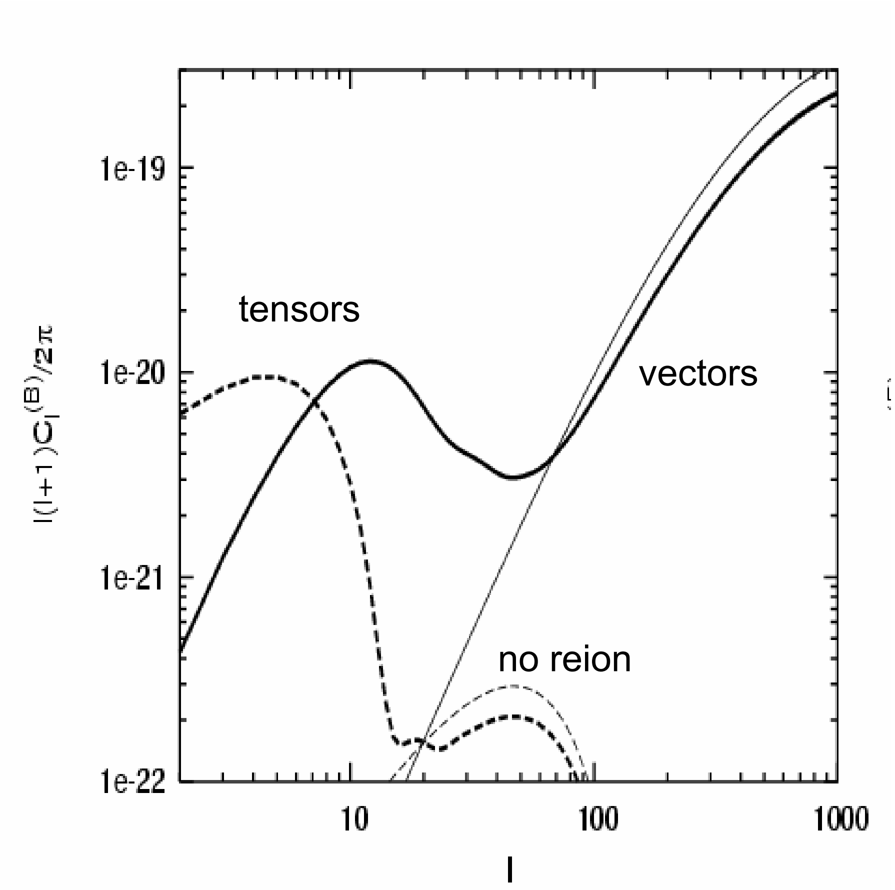
Lewis, astro-ph/0406096.
Subramanian, Seshadri, Barrow, astro-ph/0303014

- Also Faraday rotation B-modes at low frequencies

Kosowsky, Loeb: astro-ph/9601055, Scoccola, Harari, Mollerach: astro-ph/0405396

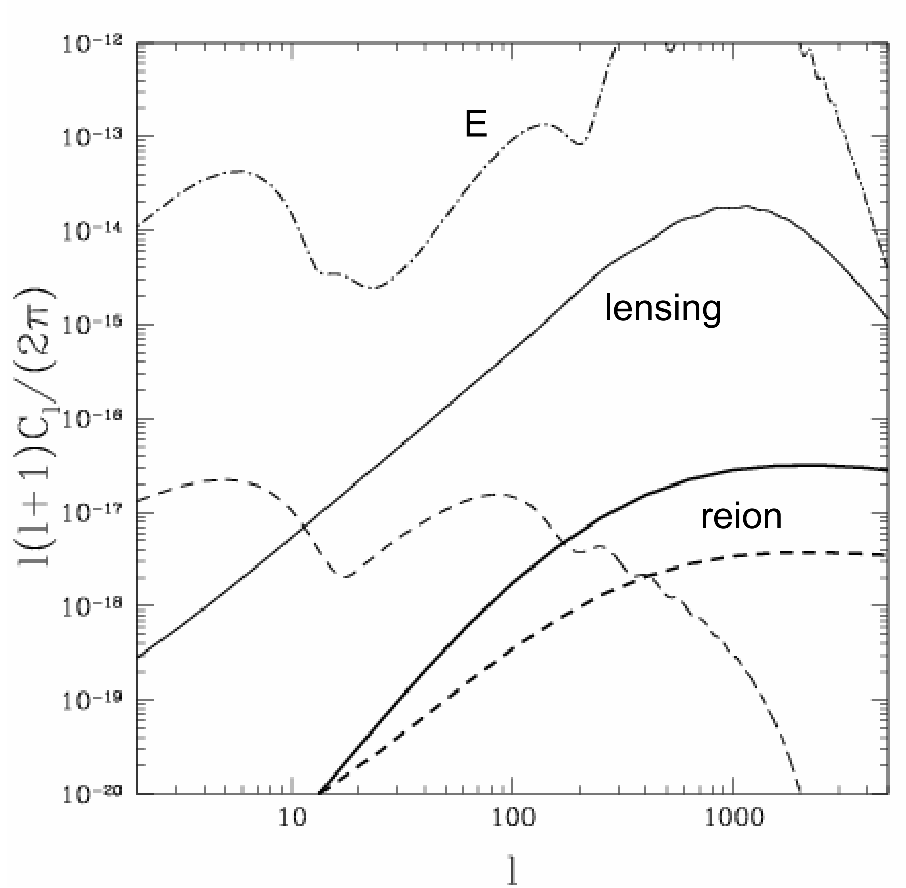
- Small second order effects, e.g.

Second order vectors and tensors:
Mollerach, Harari, Matarrese: astro-ph/0310711



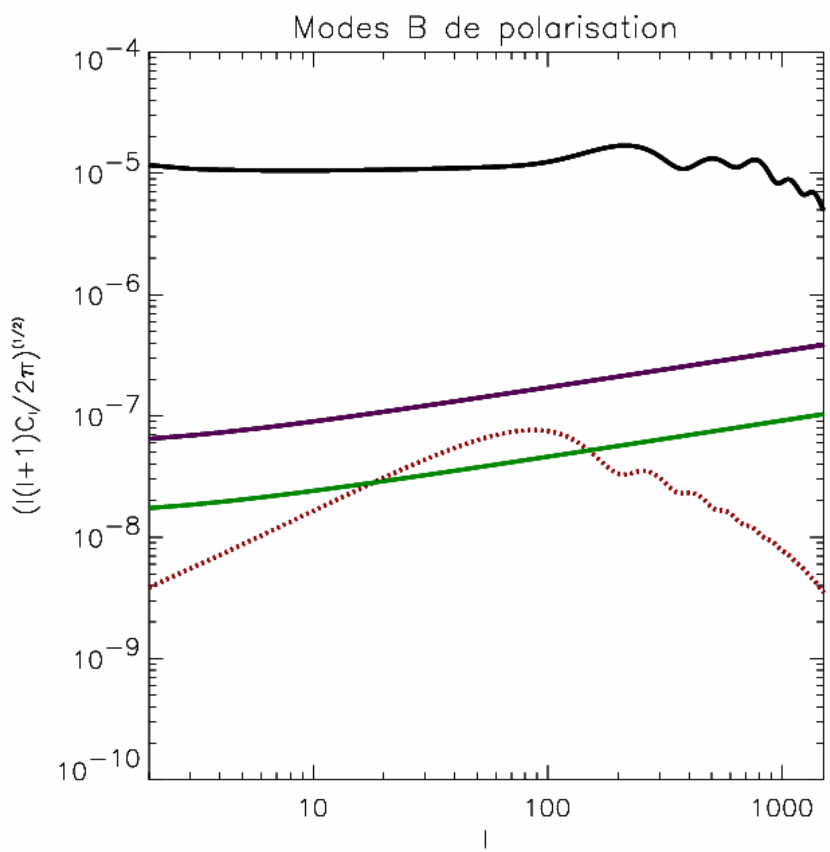
non-Gaussian

Inhomogeneous reionization
Santon, Cooray, Haiman, Knox, Ma: astro-ph/0305471; Hu: astro-ph/9907103

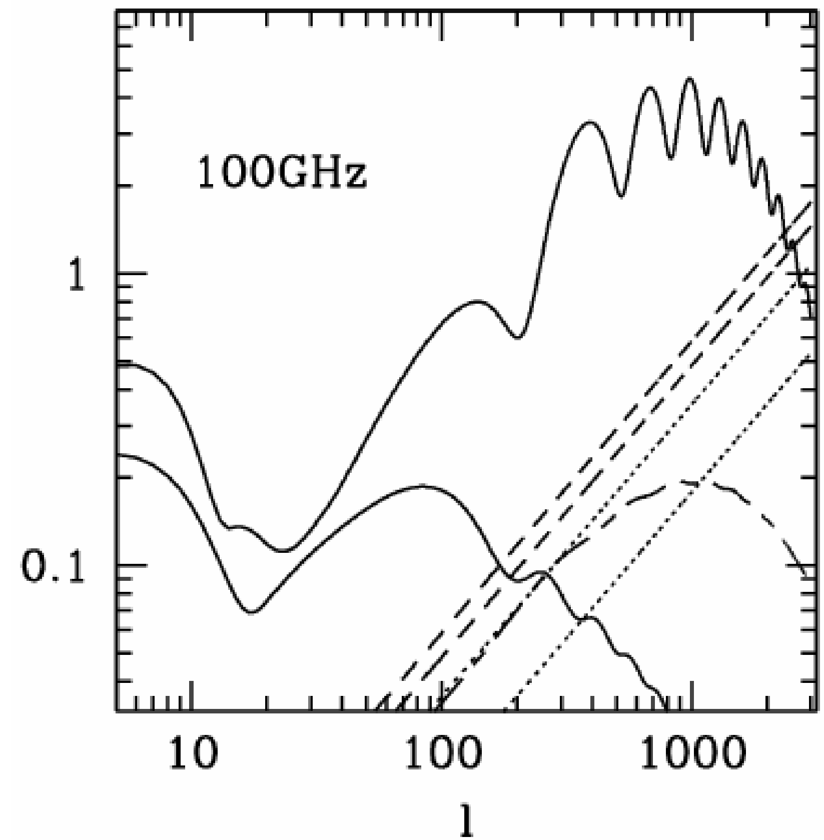


- Systematics and foregrounds, e.g.

Galactic dust (143 and 217 GHz):
Lazarian, Prunet: astro-ph/0111214



Extragalactic radio sources:
Tucci et al: astro-ph/0307073



B modes potentially a good diagnostic of foreground subtraction problems or systematics

B-mode Physics

- Large scale Gaussian B-modes from primordial gravitational waves:
  - energy scale of inflation
  - rule out most ekpyrotic and pure curvaton/ inhomogeneous reheating models and others

- non-Gaussian B-modes on small and large scales:
  - expected signal from lensing of CMB
  - other small second order signals
  - defects, magnetic fields, primordial vectors
  - foregrounds, systematics, etc.

Do we need to separate?

(to B/E or not to B/E?)

- P(sky|parameters) known, no:

  - in principle perform optimal parameter estimation without any separation, e.g. obtain $P(A_T|\text{data})$ to see whether tensor modes present

- But:

  - possible non-Gaussian signal: P(sky|parameters) unknown
  - may want robust detection of B without assumptions
  - plot C_l^{BB} for visualisation
  - map of B-modes as diagnostic, for cross-correlation, etc.
  - if signal is Gaussian, need to prove it first: separation may be first step in rigorous analysis

Cut sky E/B separation

Pure E: $$\nabla^a\nabla^b\mathcal{P}_{ab} = (\nabla^2+2)\nabla^2 P_E$$

Pure B: $$\epsilon^b{}_c\nabla^c\nabla^a\mathcal{P}_{ab} = (\nabla^2+2)\nabla^2 P_B$$

Pure B in some window function: $$B_W \equiv -2\int_S \mathrm{d}S\, W\epsilon^b{}_c\nabla^c\nabla^a\mathcal{P}_{ab}$$

Without derivatives?
Integrate by parts:

$$\begin{aligned} B_W &= \sqrt{2}\int_S \mathrm{d}S\, W_B^{ab*}\mathcal{P}_{ab} \\ &- 2\oint_{\partial S} \mathrm{d}l^a\left(\epsilon^b{}_a W\nabla^c\mathcal{P}_{cb} - \epsilon^b{}_c\nabla^c W\mathcal{P}_{ab}\right)\end{aligned}$$

→ Separation non-trivial with boundaries

- Likely important as reionization signal same scale as galactic cut

Harmonics on part of the sky

- On part of the sky harmonics are not orthogonal:

$$W_{+(lm)(lm)'} \equiv \int_S \mathrm{d}S\, W Y^{C*}_{(lm)ab} Y^{C\,ab}_{(lm)'}$$
$$W_{-(lm)(lm)'} \equiv i \int_S \mathrm{d}S\, W Y^{C*}_{(lm)ab} Y^{G\,ab}_{(lm)'}$$

- Can define cut-sky harmonic coefficients:

$$\tilde{E}_{lm} = \sqrt{2} \int \mathrm{d}S\, W Y^{G\,ab*}_{(lm)} \mathcal{P}_{ab} \qquad \tilde{B}_{lm} = \sqrt{2} \int \mathrm{d}S\, W Y^{C\,ab*}_{(lm)} \mathcal{P}_{ab}$$

$$\tilde{E}_{lm} = \sum_{(lm)'} \left(W_{+(lm)(lm)'} E_{(lm)'} + i W_{-(lm)(lm)'} B_{(lm)'} \right)$$
$$\tilde{B}_{lm} = \sum_{(lm)'} \left(W_{+(lm)(lm)'} B_{(lm)'} - i W_{-(lm)(lm)'} E_{(lm)'} \right)$$

E/B mixing

- Cut-sky harmonic coefficients mix E and B:

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{W}_+ & i\boldsymbol{W}_- \\ -i\boldsymbol{W}_- & \boldsymbol{W}_+ \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

- Pseudo-C_l mix E and B:

$$\langle \tilde{C}_l^{EE} \rangle = X_{+,ll'} C_{l'}^{EE} + X_{-,ll'} C_{l'}^{BB}$$

$$\langle \tilde{C}_l^{BB} \rangle = X_{+,ll'} C_{l'}^{BB} + X_{-,ll'} C_{l'}^{EE}$$

Pseudo-C_l / correlation function methods

- Most of the sky: Solve directly for un-mixed C_l
  Kogut et WMAP: astro-ph/0302213; Hansen and Gorski: astro-ph/0207526

  Equivalently: direct Legendre transform of correlation functions

$$C^{(EE)} = (X_+ - X_- X_+^{-1} X_-)^{-1} \left\langle \tilde{C}^{EE} - X_- X_+^{-1} \tilde{C}^{BB} \right\rangle$$

$$C^{(BB)} = (X_+ - X_- X_+^{-1} X_-)^{-1} \left\langle \tilde{C}^{BB} - X_- X_+^{-1} \tilde{C}^{EE} \right\rangle$$

- Bit of the sky: Coupling matrix is singular, inversion impossible

  Equivalently: Only part of correlation function so not Legendre invertible

  - Construct combination of correlation function integrals to remove mixing: Polarized SPICE

  Crittenden, Natarajan, Pen, Theuns: astro-ph/0012336
  Chon, Challinor, Prunet, Hivon, Szapudi :astro-ph/0303414

Properties

- Can achieve exact E/B power spectrum separation **on average**

- In any given realization some mixing: **feels cosmic variance**

  e.g. estimator is

$$\hat{C}^{BB} = (X_+ - X_- X_+^{-1} X_-)^{-1} \left(\tilde{C}^{BB} - X_- X_+^{-1} \tilde{C}^{EE} \right)$$

$$\left\langle \hat{C}^{BB} \right\rangle = C^{BB}$$ but $$\hat{C}^{BB} \neq 0 \quad \not\Rightarrow \quad C^{BB} \neq 0$$

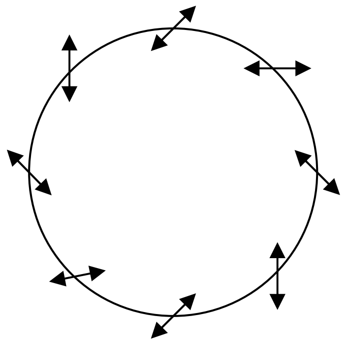
If B=0 $$\left\langle \hat{C}^{BB} \hat{C}^{BB\dagger} \right\rangle = f(C^{EE})$$

- Fast, practical method for estimating separated power spectra: cosmic variance << noise for near future

- Only measures 2-point information, no guide to map-level separation

Map level E/B separation

- Circles on the sky



In polar co-ordinates $$\hat{B} = \oint dl\, U$$

No cosmic variance: $$\hat{B} \neq 0 \quad \Longrightarrow \quad B \neq 0$$

About one axis: extracts some of the information, 'm=0' modes only
About every possible axis: complicated, not independent

Chiueh, Ma: astro-ph/0101205; Zaldarriaga: astro-ph/0106174

- General methods

Lewis: astro-ph/0305545
Bunn, Zaldarriaga, Tegmark, de Oliveira-Costa: astro-ph/0207338
Lewis, Challinor, Turok: astro-ph/0106536

Real space measures

$$B_W = \sqrt{2}\int_S \mathrm{d}S\, W_B^{ab*}\mathcal{P}_{ab} - 2\oint_{\partial S} \mathrm{d}l^a \left(\epsilon^b{}_a W \nabla^c \mathcal{P}_{cb} - \epsilon^b{}_c \nabla^c W \mathcal{P}_{ab}\right)$$

- Can measure B without derivatives or line integrals by taking window W so that

$$W = 0 \qquad \nabla W = 0 \quad \text{on } \partial S$$

(in general)

- Find complete set of window functions: extract all the B information

Lewis, Challinor, Turok: astro-ph/0106536
Bunn, Zaldarriaga, Tegmark, de Oliveira-Costa: astro-ph/0207338

General harmonic separation

- Extract pure E and B modes from observed cut sky

$$\begin{pmatrix}\tilde{\mathrm{E}}\\ \tilde{\mathrm{B}}\end{pmatrix} = \begin{pmatrix} \boldsymbol{W}_+ & i\boldsymbol{W}_- \\ -i\boldsymbol{W}_- & \boldsymbol{W}_+ \end{pmatrix} \begin{pmatrix}\mathrm{E}\\ \mathrm{B}\end{pmatrix}$$

- No band limit: optimal result for B is

if **P**$_-$ projects out the range of **W**$_-$: $\boldsymbol{P}_-\boldsymbol{W}_- = 0$

$$\hat{\mathrm{B}} = \boldsymbol{P}_-\tilde{\mathrm{B}} = \begin{pmatrix}\mathbf{0} & \boldsymbol{P}_-\end{pmatrix} \begin{pmatrix} \boldsymbol{W}_+ & i\boldsymbol{W}_- \\ -i\boldsymbol{W}_- & \boldsymbol{W}_+ \end{pmatrix} \begin{pmatrix}\mathrm{E}\\ \mathrm{B}\end{pmatrix}$$

$$= \boldsymbol{P}_-\boldsymbol{W}_+\mathrm{B}$$

P$_-$ can be constructed explicitly by SVD methods (astro-ph/0106536)

Solution for separation matrix

Want $\boldsymbol{P}_- \boldsymbol{W}_- = 0$

$$\boldsymbol{W}_- = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger$$

$$\boldsymbol{U}^\dagger \boldsymbol{W}_- = \boldsymbol{D}\boldsymbol{U}^\dagger$$

$$\boldsymbol{D}'\boldsymbol{U}^\dagger \boldsymbol{W}_- = 0 \quad \text{where} \quad \boldsymbol{D}'\boldsymbol{D} = 0$$

$\mathbf{D}_{ii}'$ **=** 1 [$\mathbf{D}_{ii}$ = 0]
0 [otherwise]

Separation matrix is $\boldsymbol{D}'\boldsymbol{U}^\dagger$

$$\hat{\mathrm{B}} = \boldsymbol{D}'\boldsymbol{U}^\dagger \tilde{\mathrm{B}} = \boldsymbol{D}'\boldsymbol{U}^\dagger \boldsymbol{W}_+ \mathrm{B}$$

Properties

- **W₋** is a boundary integral
  - equivalent to projecting out line integrals

$$B_W = \sqrt{2}\int_S \mathrm{d}S\, W_B^{ab*}\mathcal{P}_{ab} - 2\oint_{\partial S} \mathrm{d}l^a \left(\epsilon^b{}_a W \nabla^c \mathcal{P}_{cb} - \epsilon^b{}_c \nabla^c W \mathcal{P}_{ab}\right)$$

- `Ambiguous' modes: residuals have E and B, cannot be separated

- Optimal: extracts all pure B information there is

- Slow: requires diagonalization of l_{max}^2 x l_{max}^2 matrix in general
  – computationally impossible unless azimuthally symmetric

- No assumptions about statistics: can test for Gaussianity etc.

Practical method

- Most of B from gravity waves on large scales l < 300
  for high optical depth most from l < 30
- But also E signal on much smaller scales

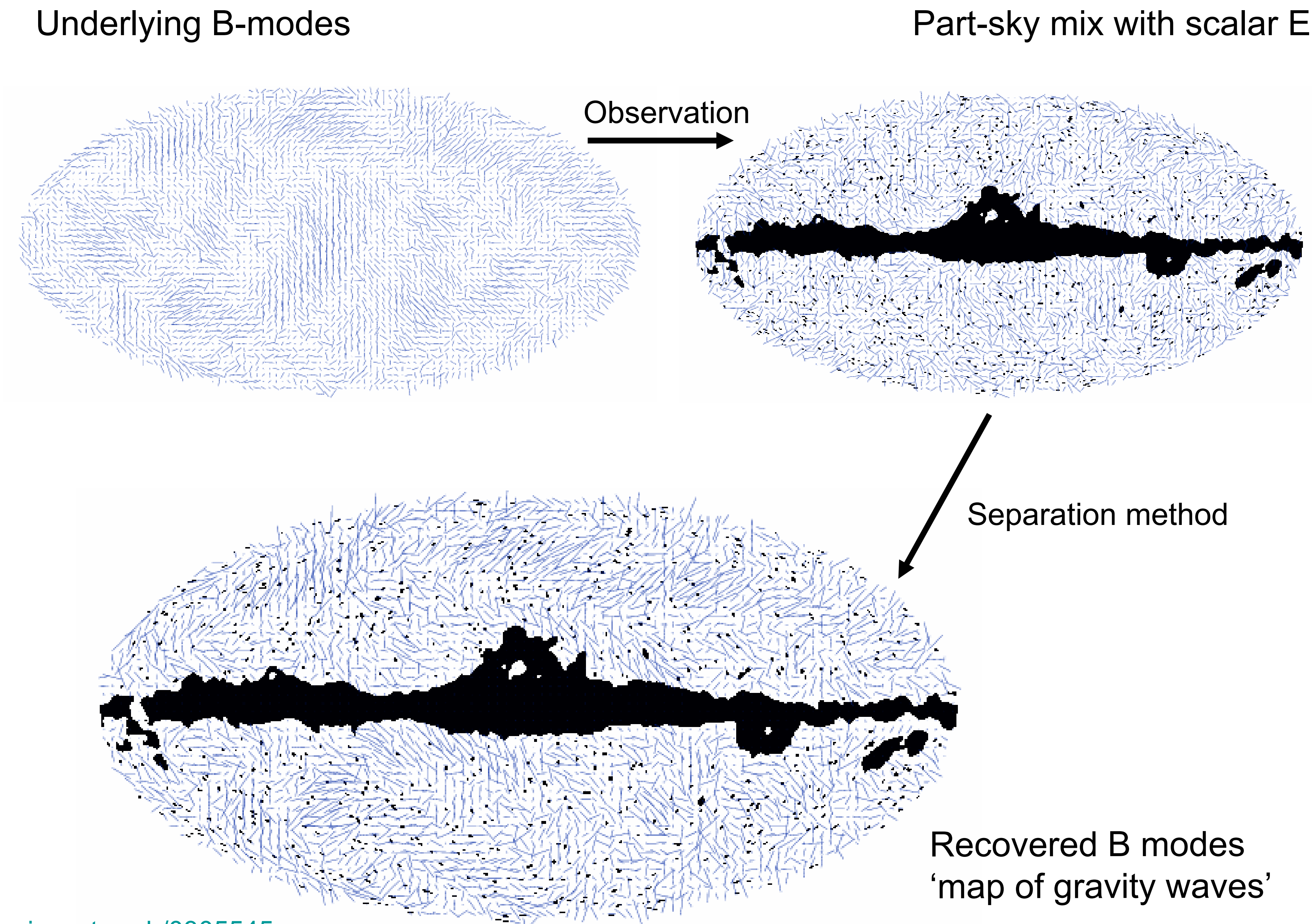
  - Impose low band limit by convolution, increase cut size correspondingly

  - More systematic method: use well supported modes

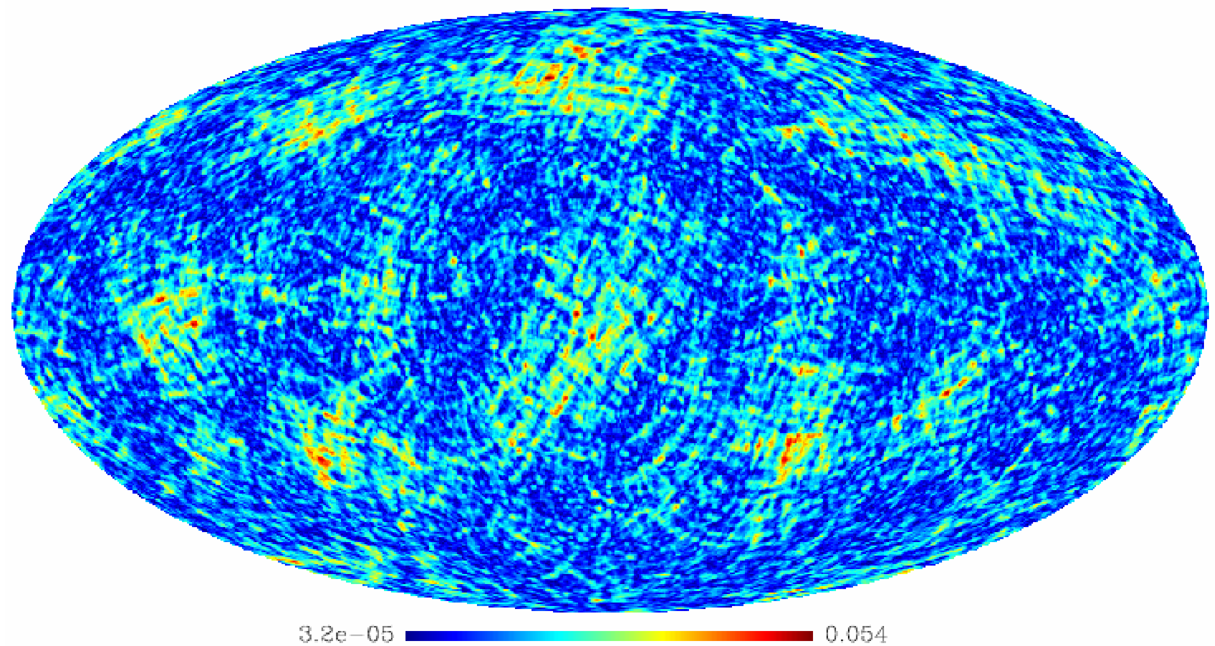
  if $\boldsymbol{W}_{+}\mathrm{e} = (1-\epsilon)\mathrm{e}$ then $\mathrm{e}^{\dagger}\tilde{\mathbf{B}}$ is nearly pure B

  Lewis: astro-ph/0305545

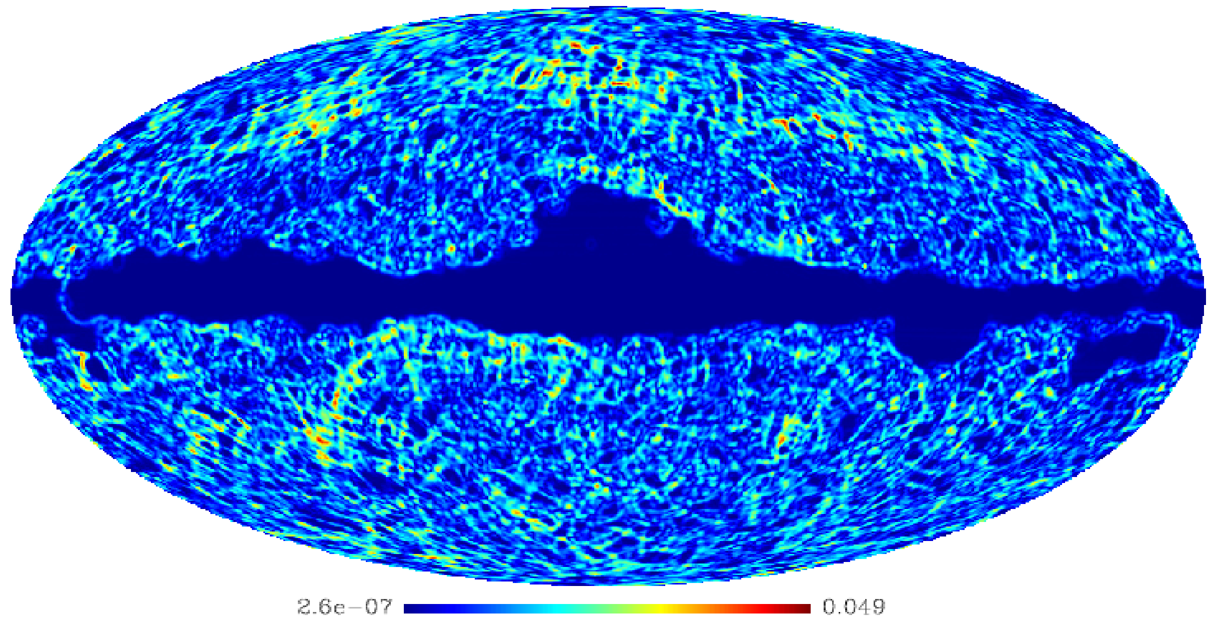
- Diagonalization computationally 'tractable' for l_{max} < 300
  or use conjugate gradients

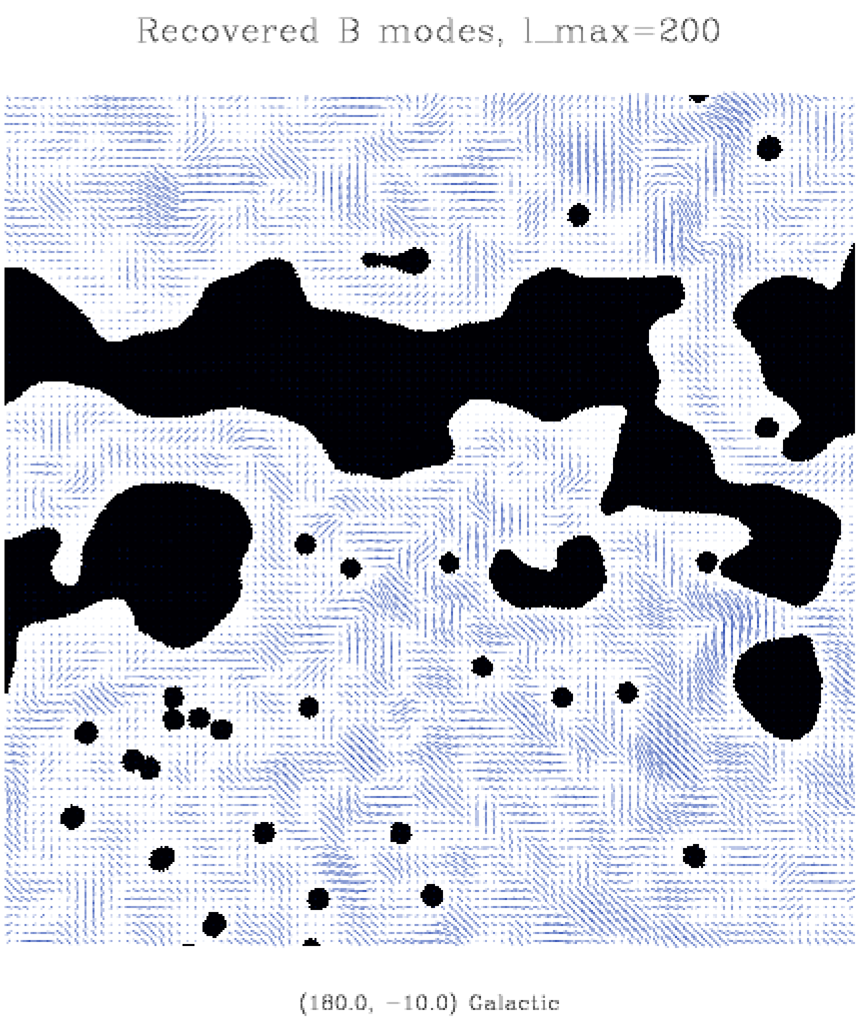


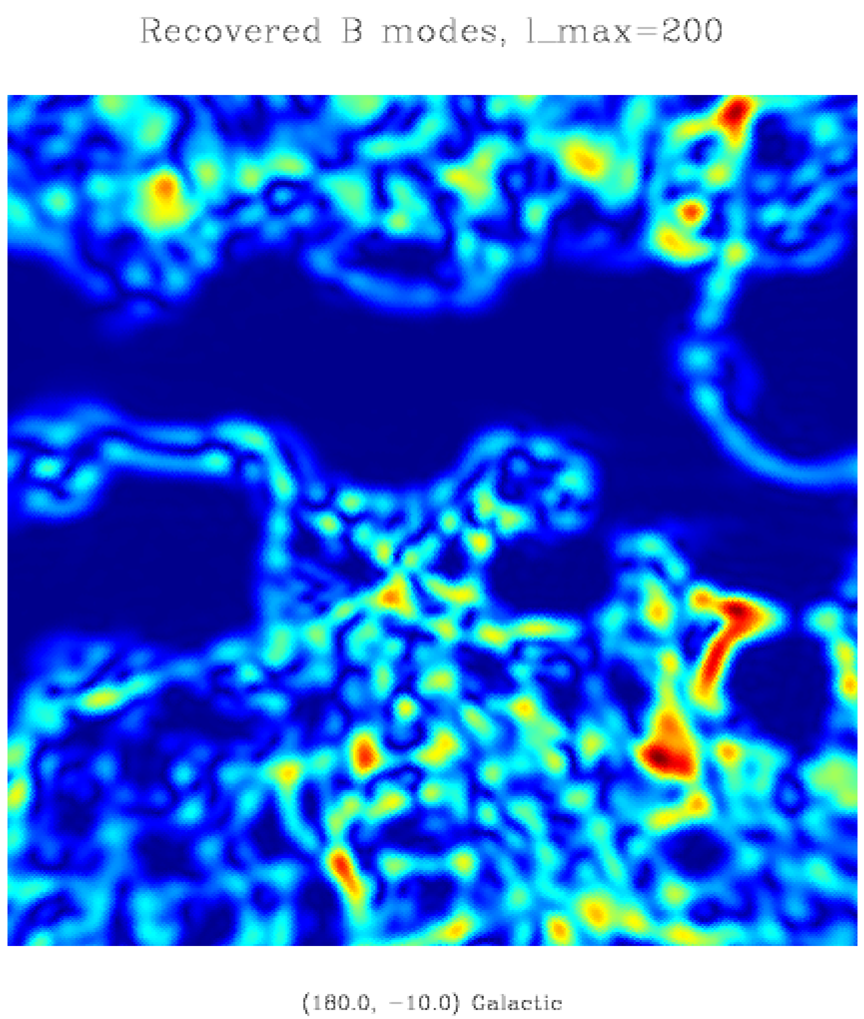
Lewis: astro-ph/0305545



Magnitude of recovered B polarization,l_max=200







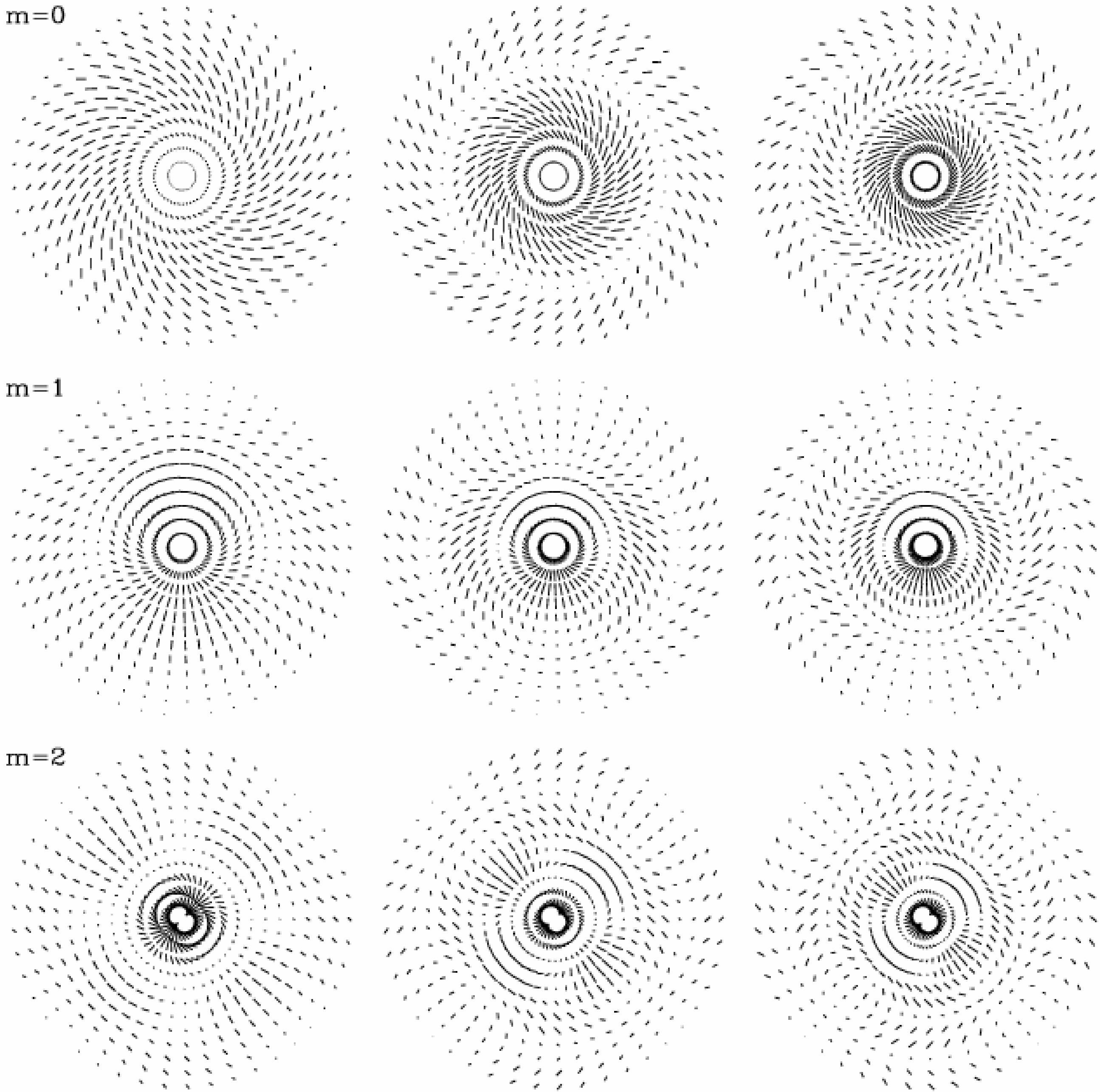
http://cosmologist.info/polar/EBsupport.html

Conclusions

- Lots of interesting things to be learnt from B-modes
- E/B separation trivial on the full sky
- Separation non-local: mode and C_l mixing on the cut sky

- Power spectrum separation methods simple and fast
  - cosmic variance, but smaller than noise for near future

- Map level separation methods
  - extracts all the B-mode information, not just 2-point
  - no assumptions about statistics (test for Gaussianity)
  - practical nearly exact computationally tractable methods available (on large scales; on all scales if azimuthal symmetry)
  - possible visual diagnostic of systematics etc.



Lewis, Challinor, Turok: astro-ph/0106536