4.5 Post-Newtonian order gravitational radiation

Tanguy Marchand - IAP
marchand@iap.fr

T. Marchand, L. Blanchet, G. Faye
arxiv:1607.07601 (submitted to QCG)

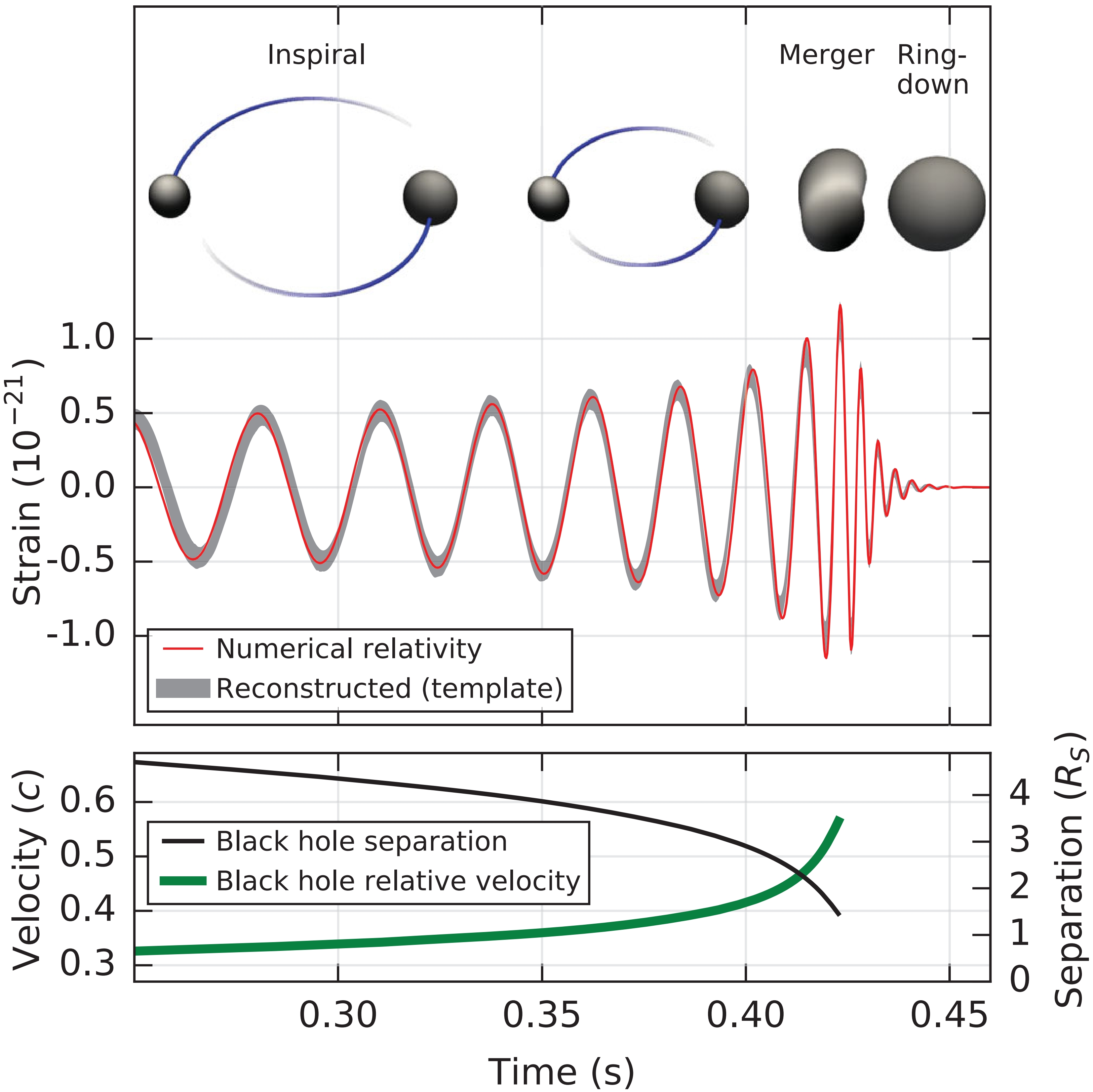
GRAMPA meeting - 30 august 2016

I. Introduction

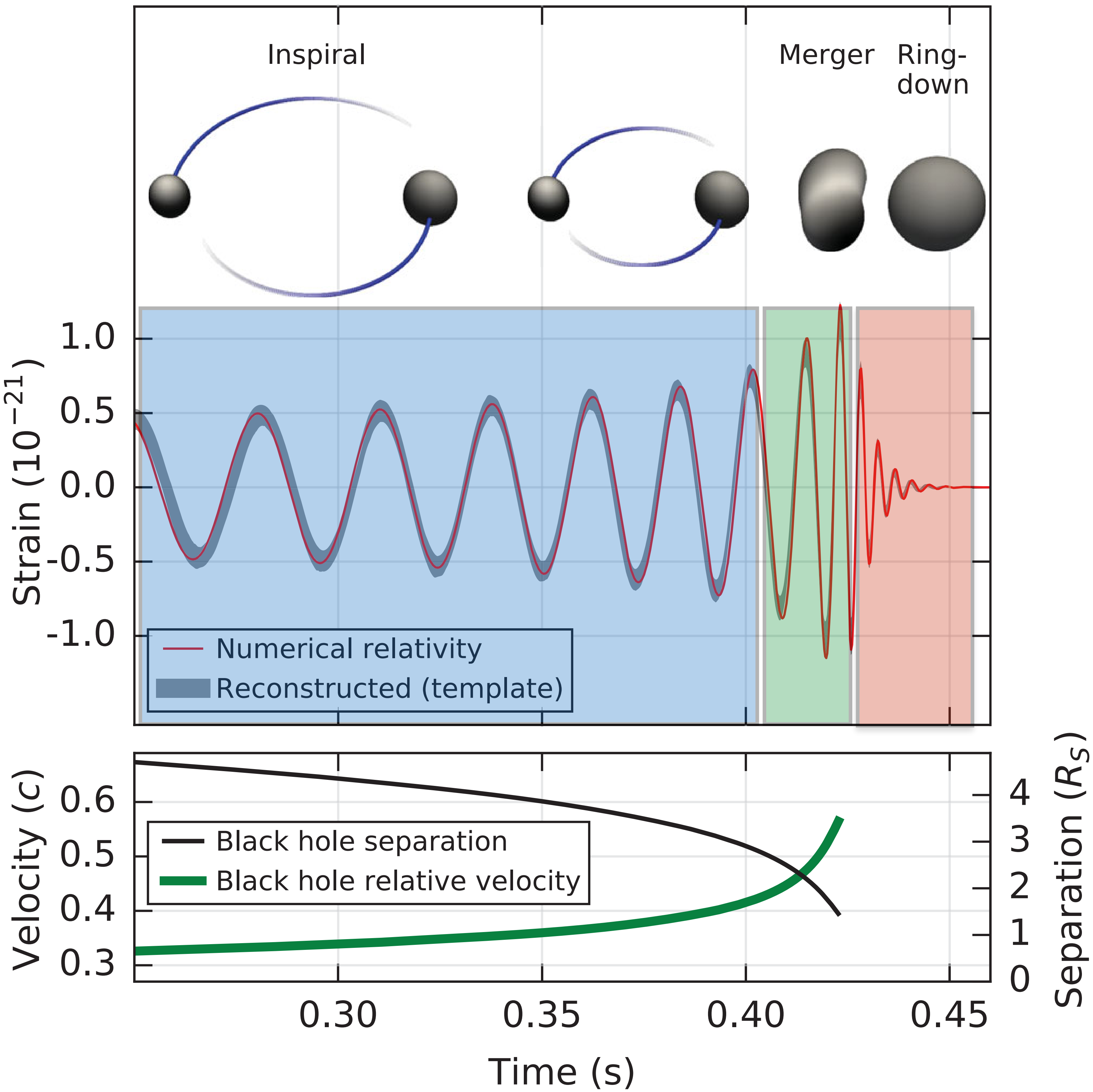
II. The multipolar-post-Minkowskian algorithm

III. The 4.5PN project

I. INTRODUCTION



PRL 116, 061102 (2016)



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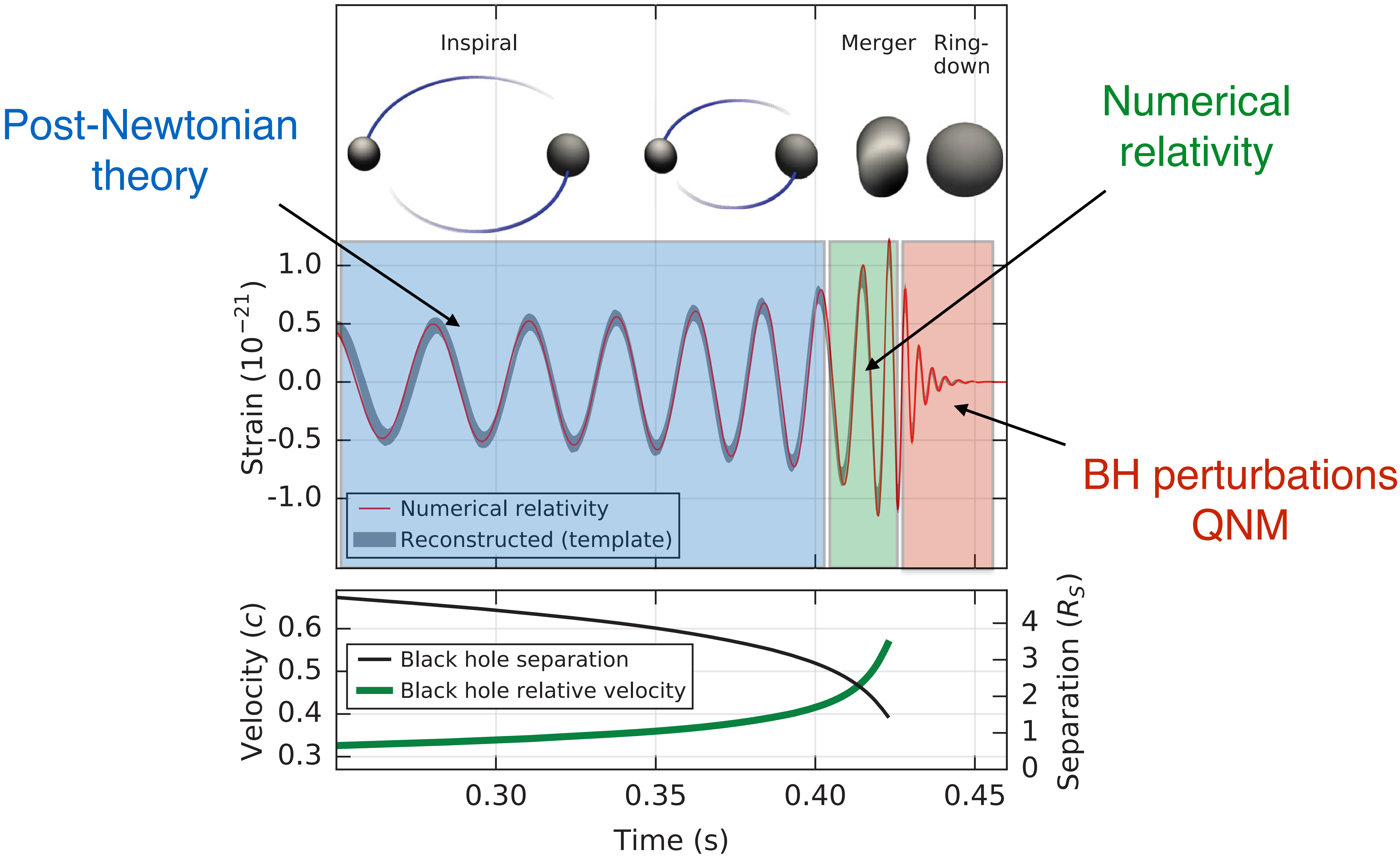

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Post-Newtonian theory

- Perturbative expansion of relativistic effects

- 1 PN $\longrightarrow \left(\frac{v}{c}\right)^2$

- More and more difficulties appear as we go to higher orders

Blanchet-Damour-Iyer formalism

Blanchet-Damour-Iyer formalism

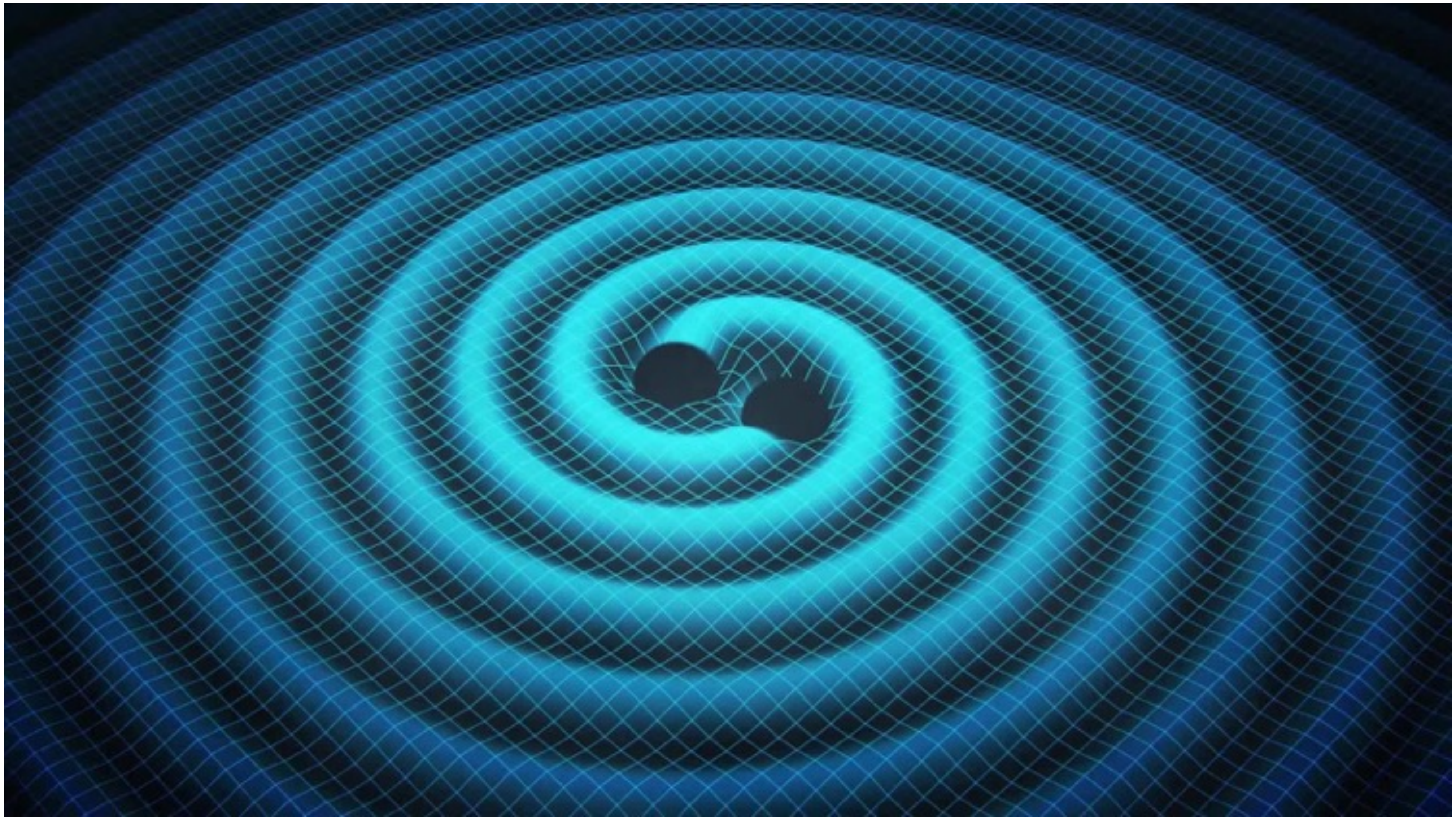
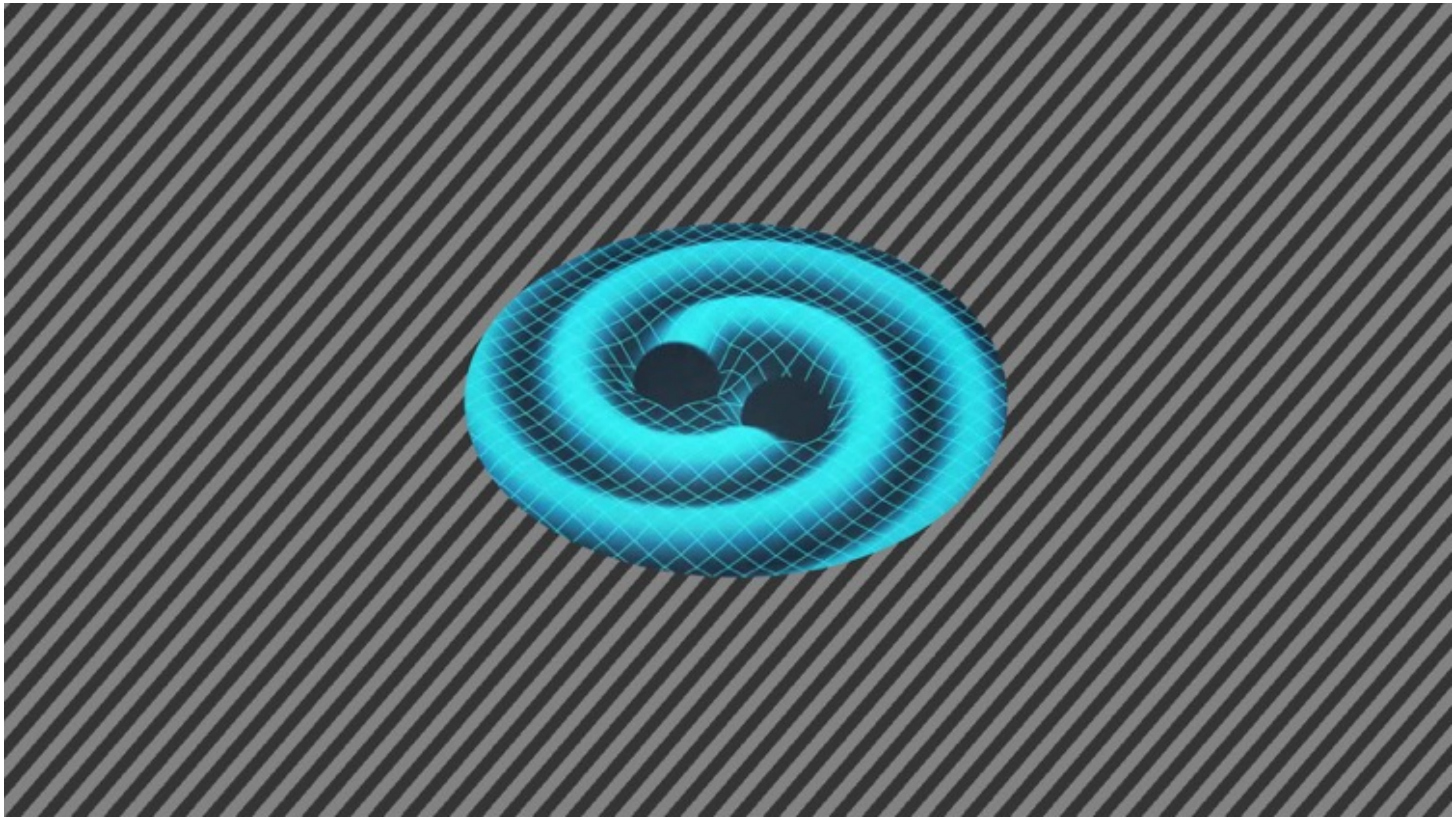


figure: www.virgo-gw.eu

Blanchet-Damour-Iyer formalism



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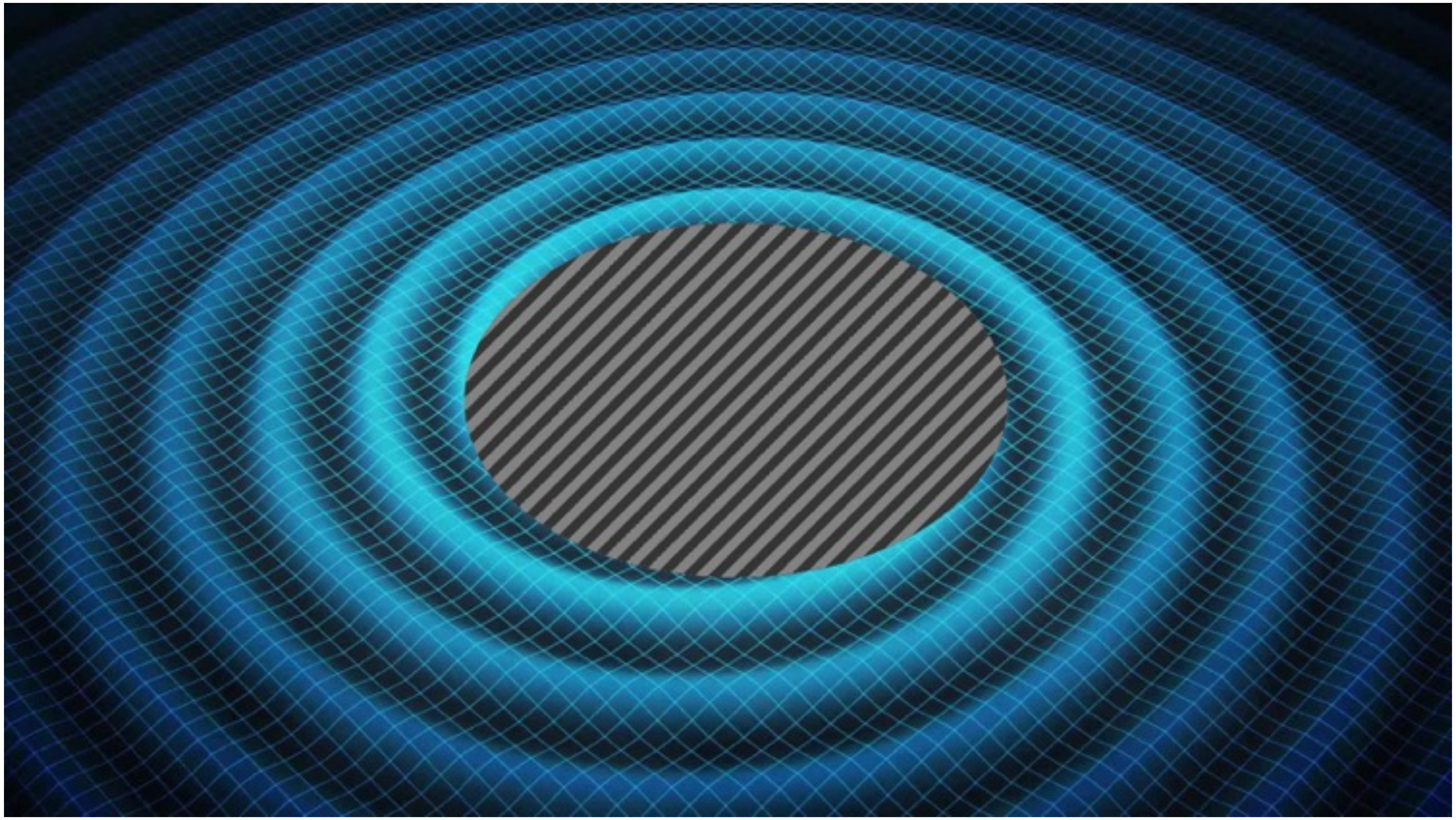

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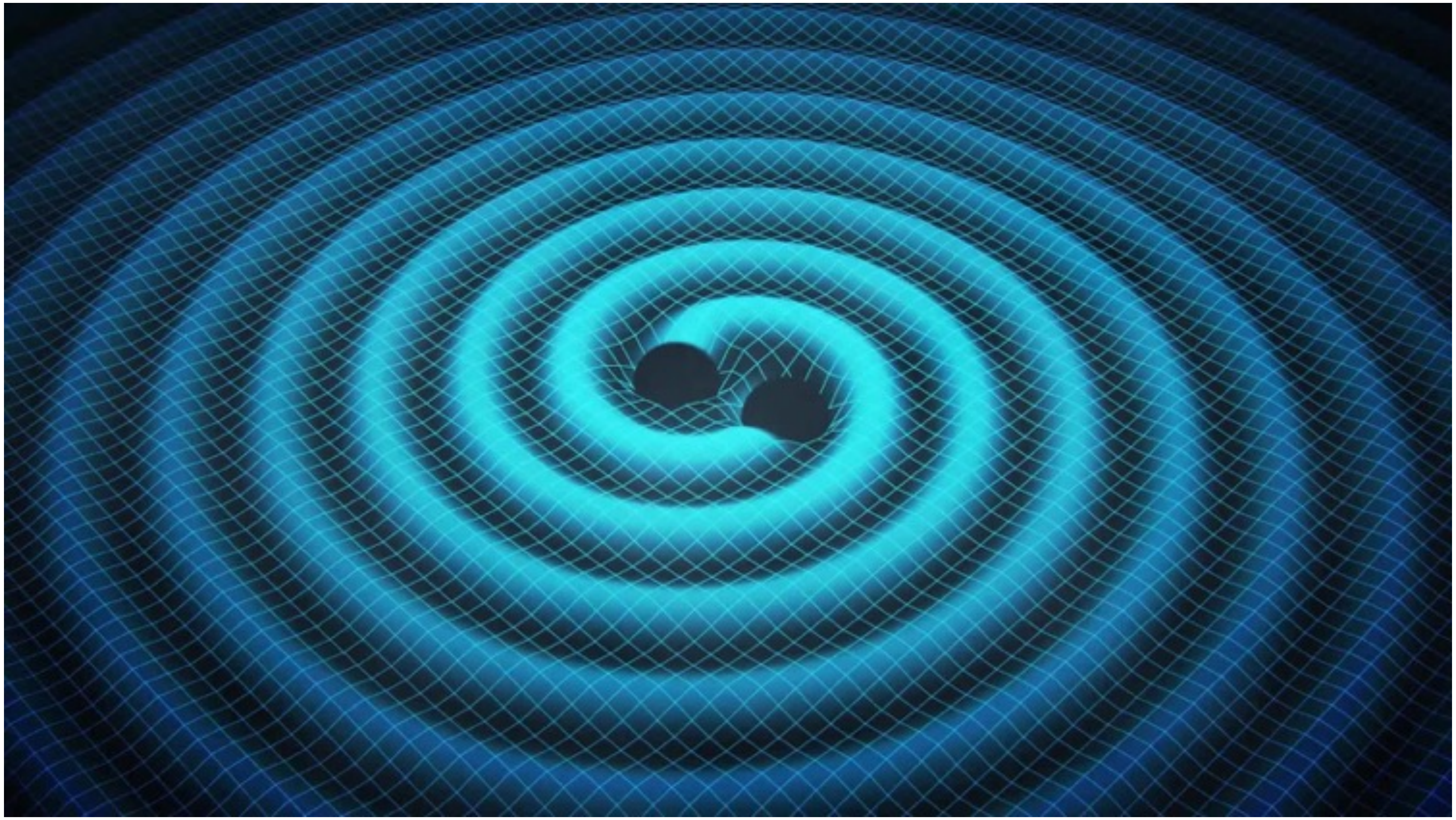

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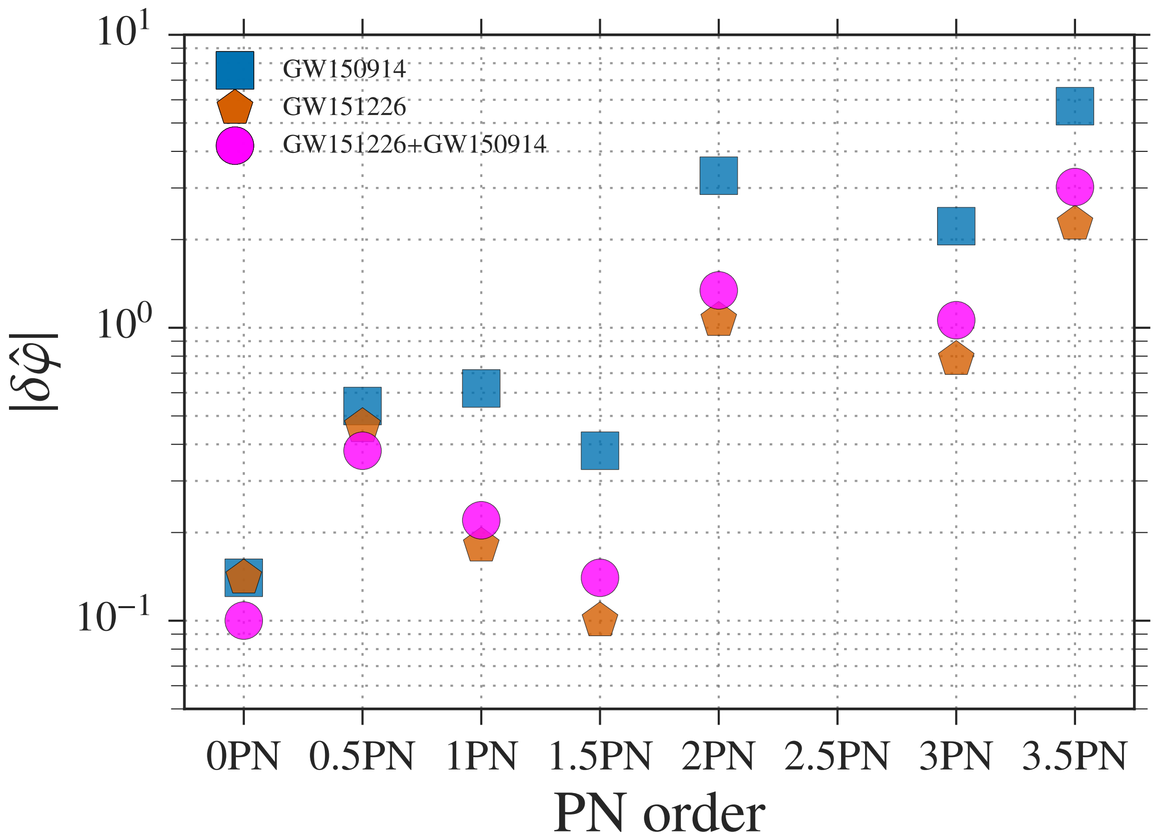
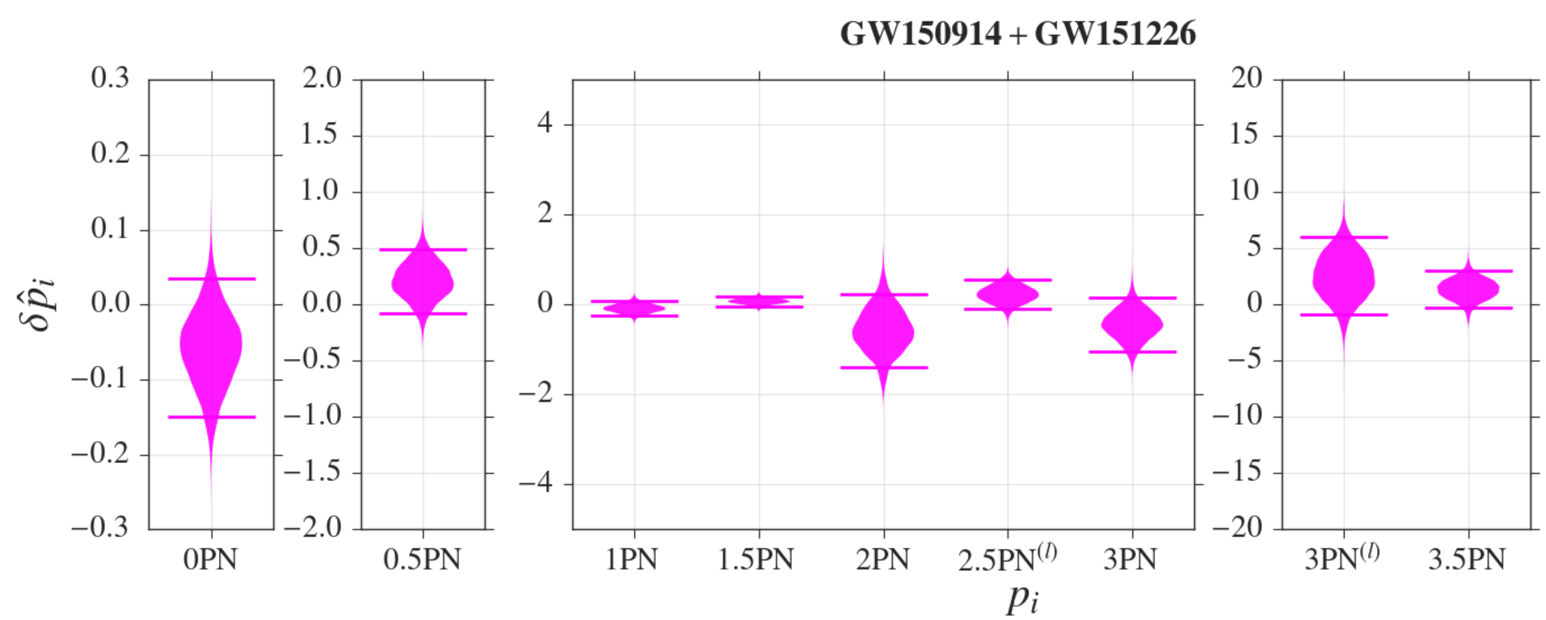
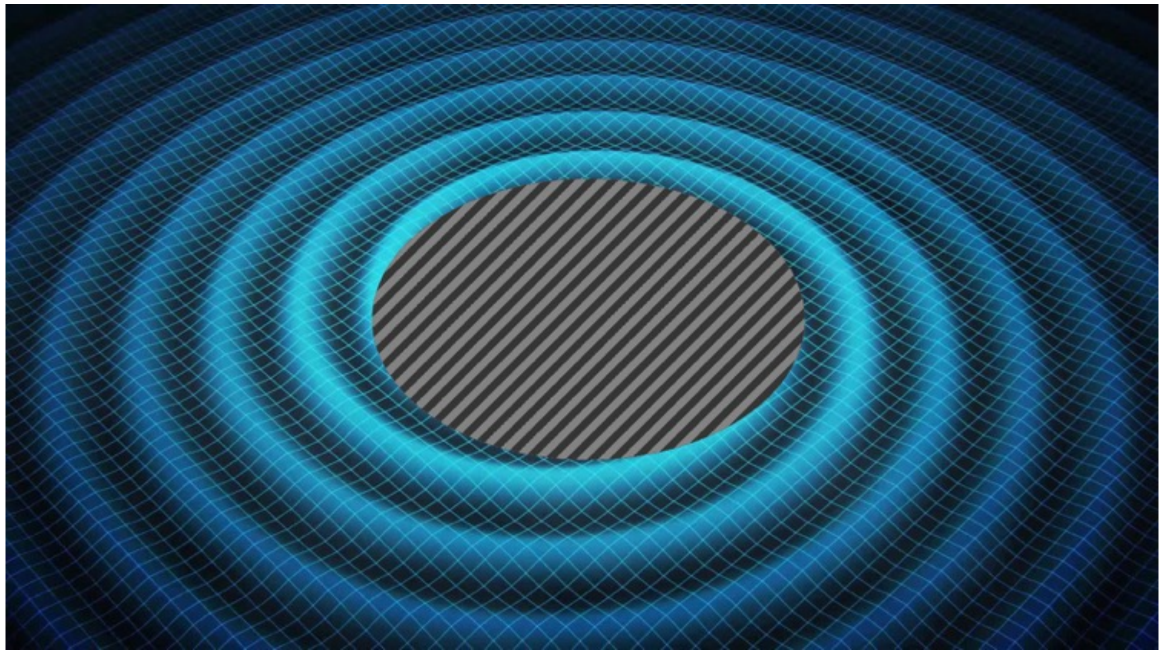




LIGO Scientific and Virgo collaboration arxiv:1606.04856

II. The multipolar post-Minkowskian (MPM) algorithm

The MPM algorithm



$$G_{\mu\nu}(g_{\alpha\beta}, \partial g_{\alpha\beta}, \partial^2 g_{\alpha\beta}) = 0$$

$$h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu} = \mathcal{G} h^{1\mu\nu} + \mathcal{G}^2 h^{2\mu\nu} + \ldots$$

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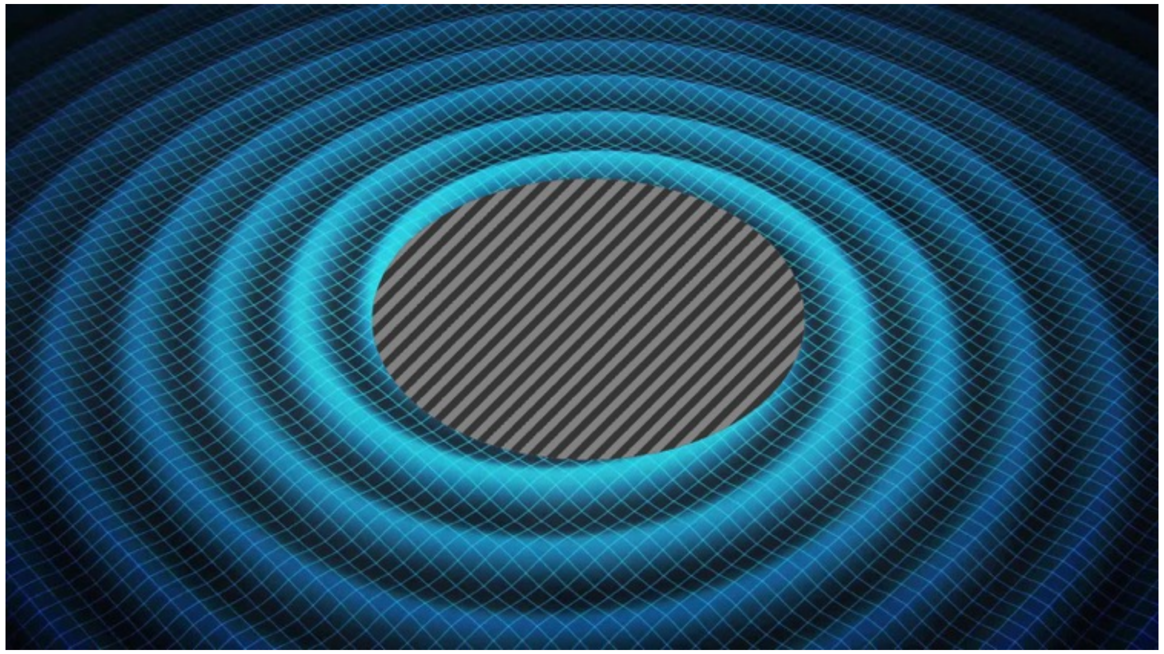

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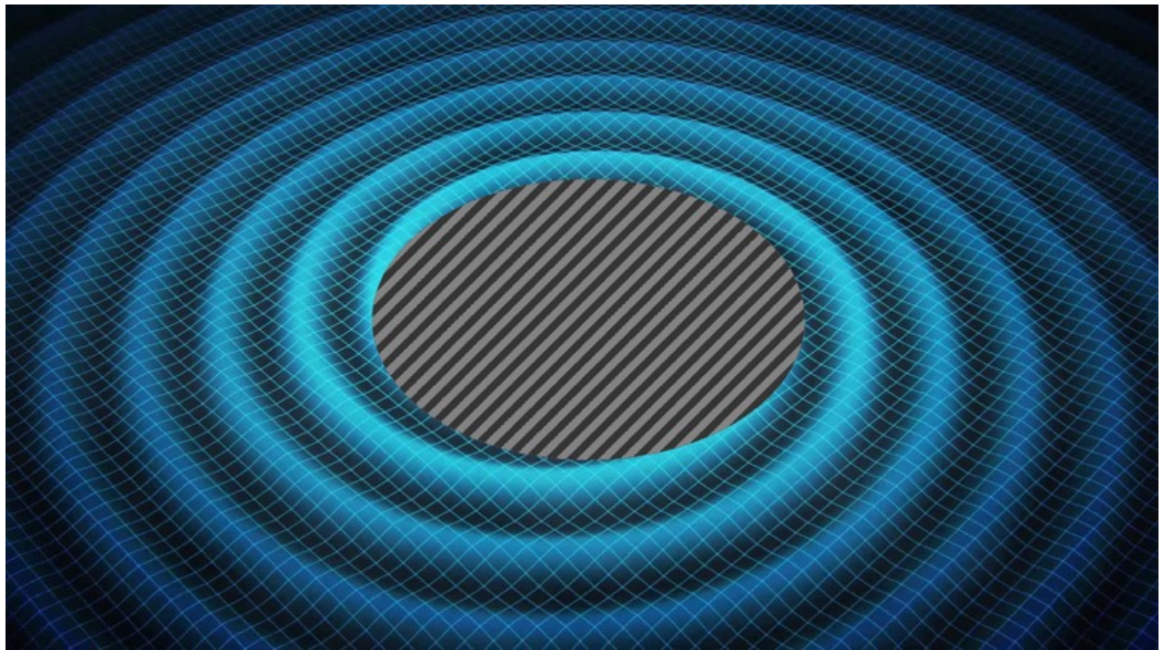

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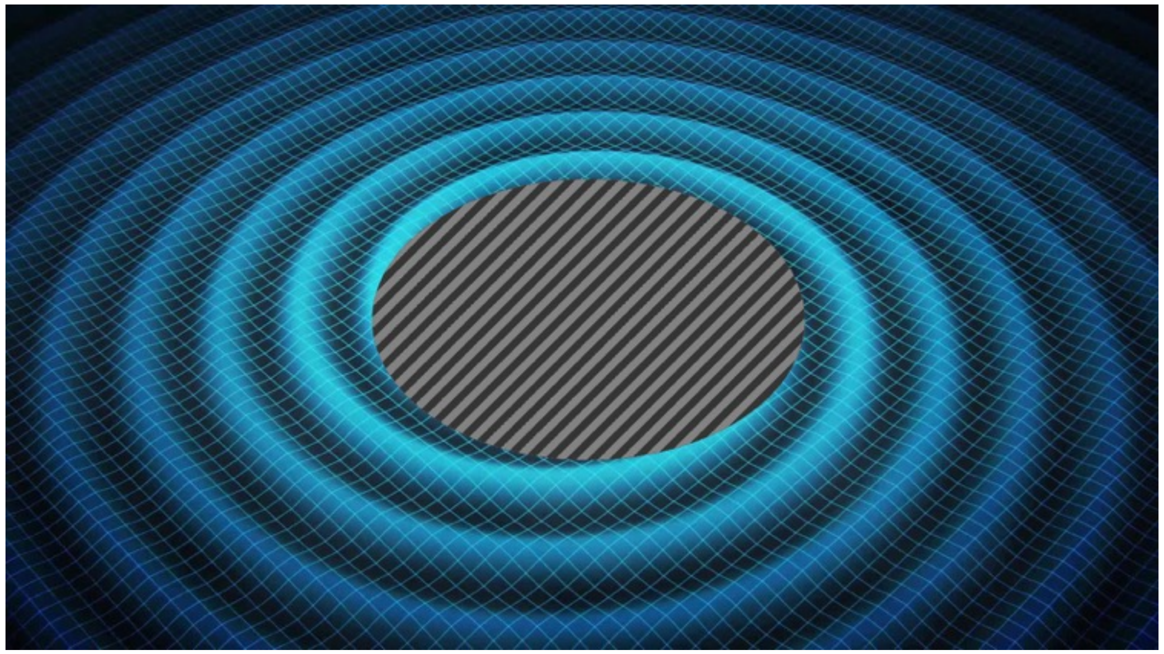

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$$h^1_{\mu\nu} \sim \sum_{l\geq 0} \partial_{i_1,\ldots,i_l}\left(\frac{M_{i_1\ldots i_l}(t-r)}{r}\right) + \sum_{l\geq 2} \partial_{i_1,\ldots,i_l}\left(\frac{S_{i_1\ldots i_l}(t-r)}{r}\right)$$
$$= h^1_M + h^1_{M_{ij}} + h^1_{M_{ijk}} + \cdots + h^1_{S_{ij}} + h^1_{S_{ijk}} + \ldots$$

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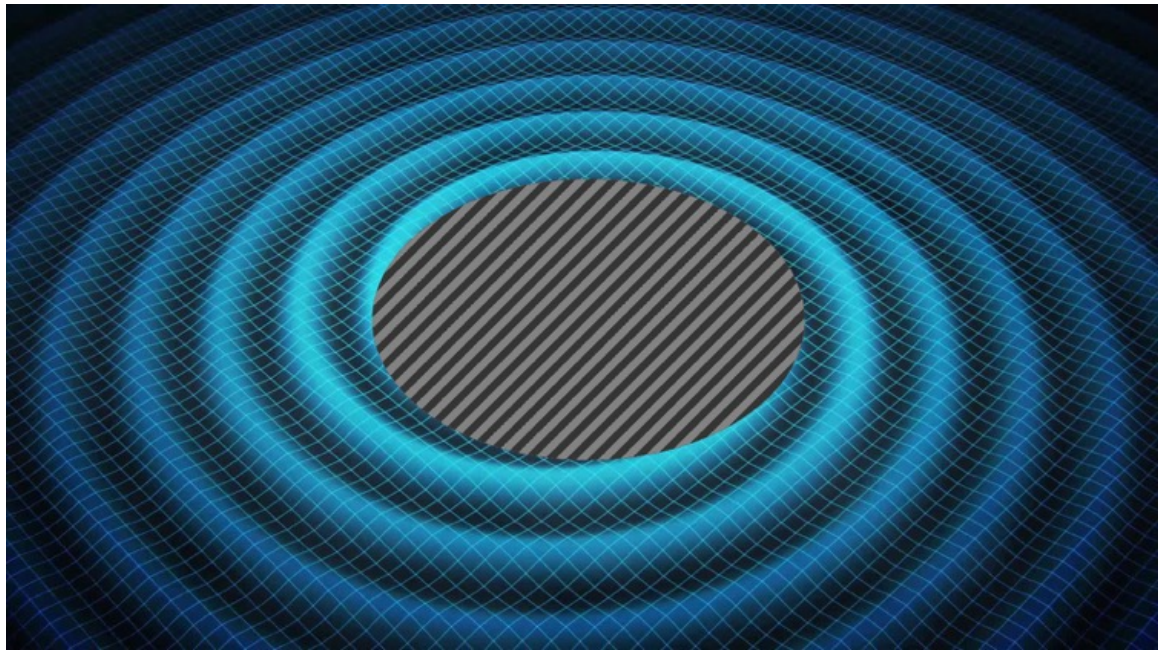

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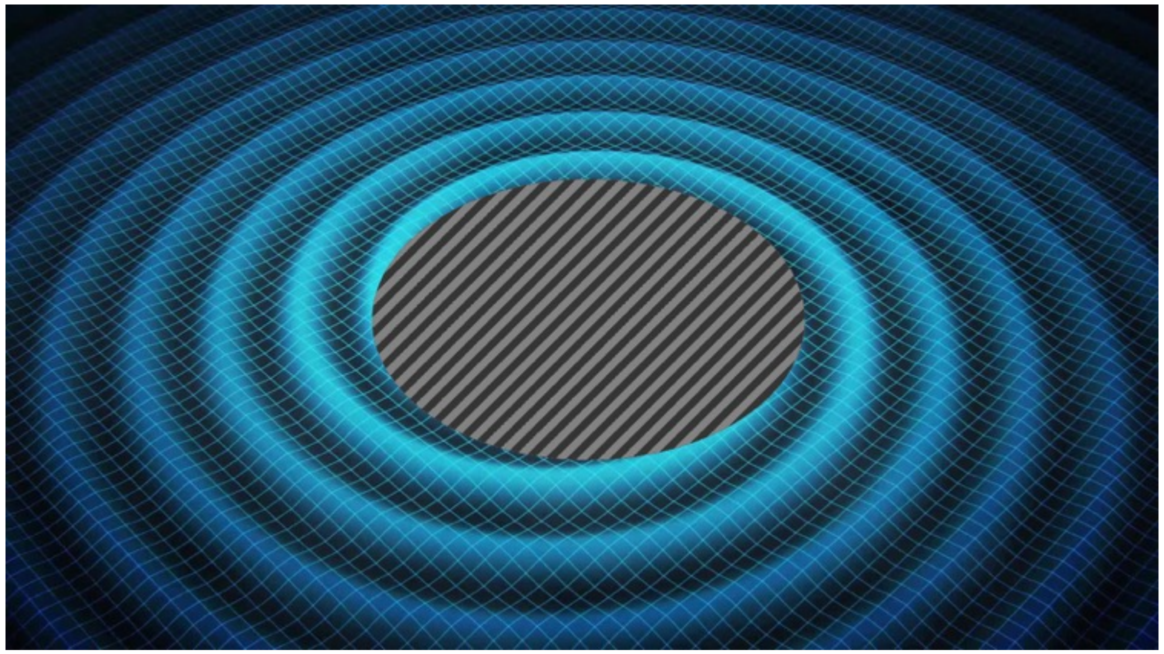
$$h^2_{\mu\nu} = h^2_{M\times M} + h^2_{M\times M_{ij}} + h^2_{M_{ij}\times M_{ij}} + \ldots$$

First issue: regularization



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Issue: $\Lambda \sim_{r\to 0} \frac{1}{r^k}, \; k \geq 3$

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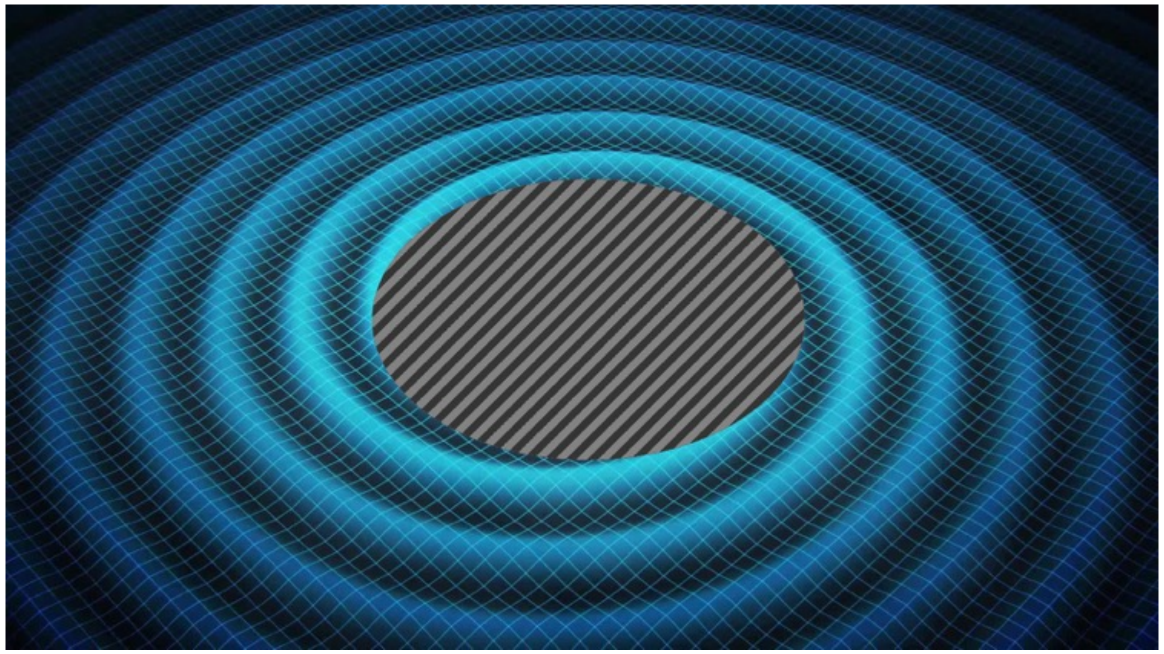

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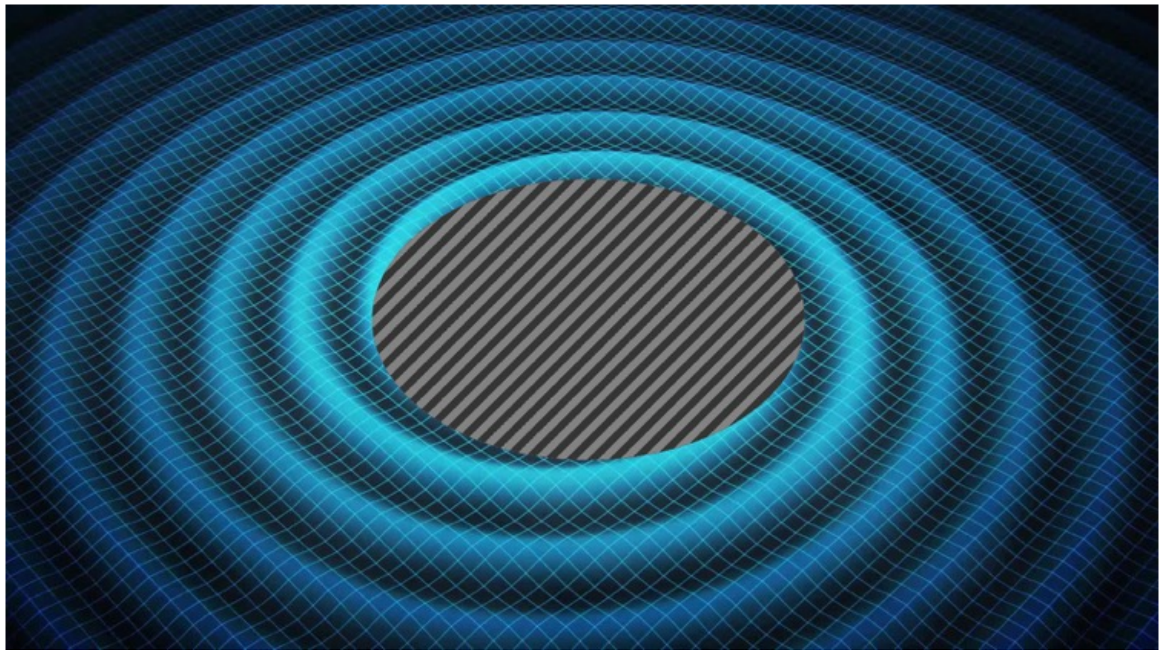

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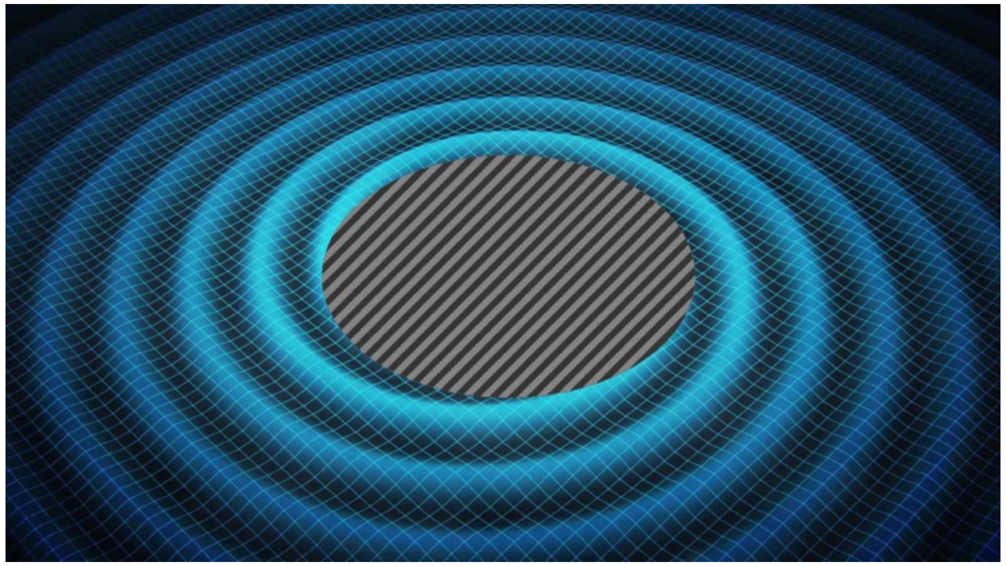
$$\mathrm{FP}_{B=0}\left[\sum_{k\geq -k_0} g_k B^k\right] \equiv g_0$$

Second issue: tails



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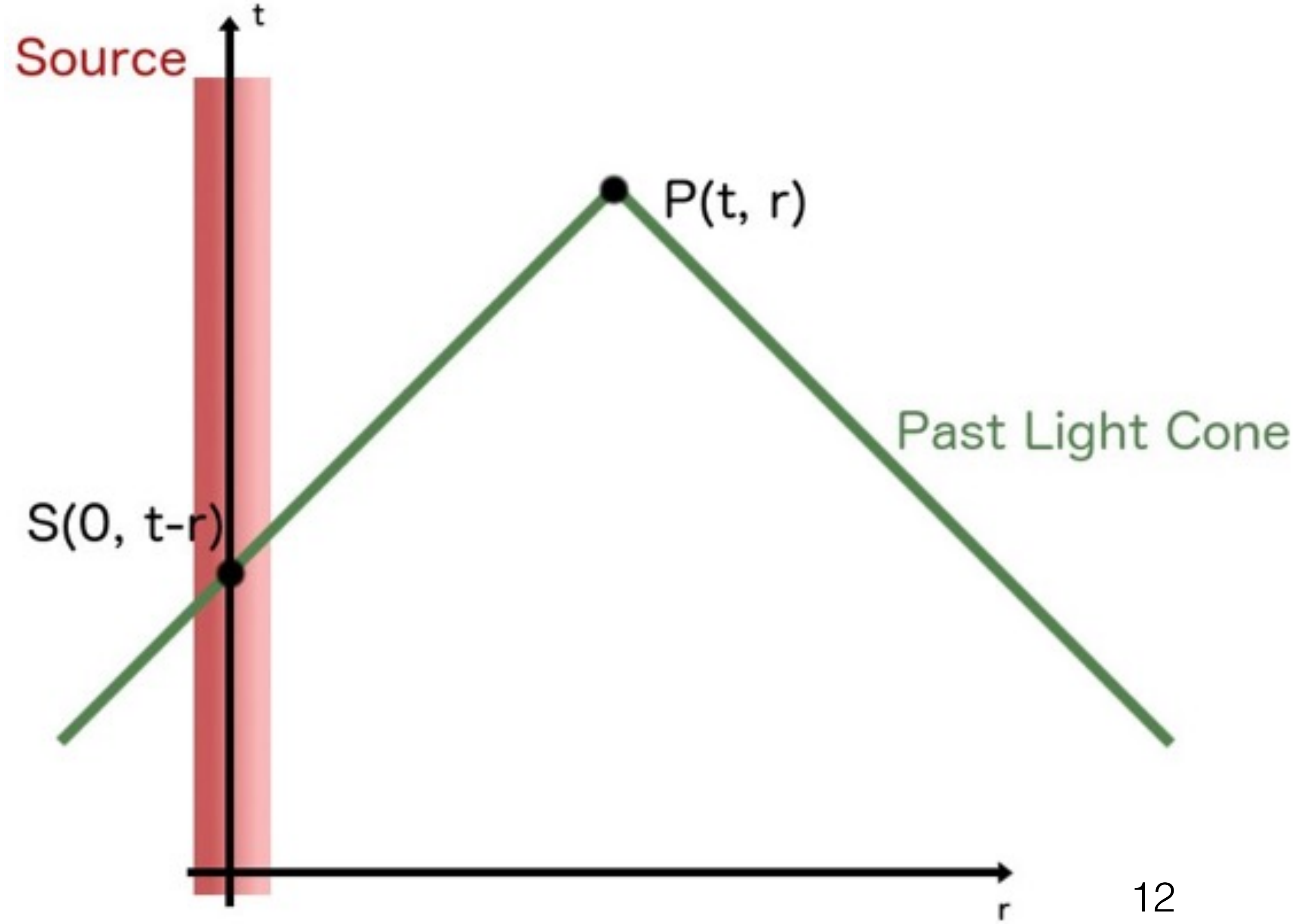

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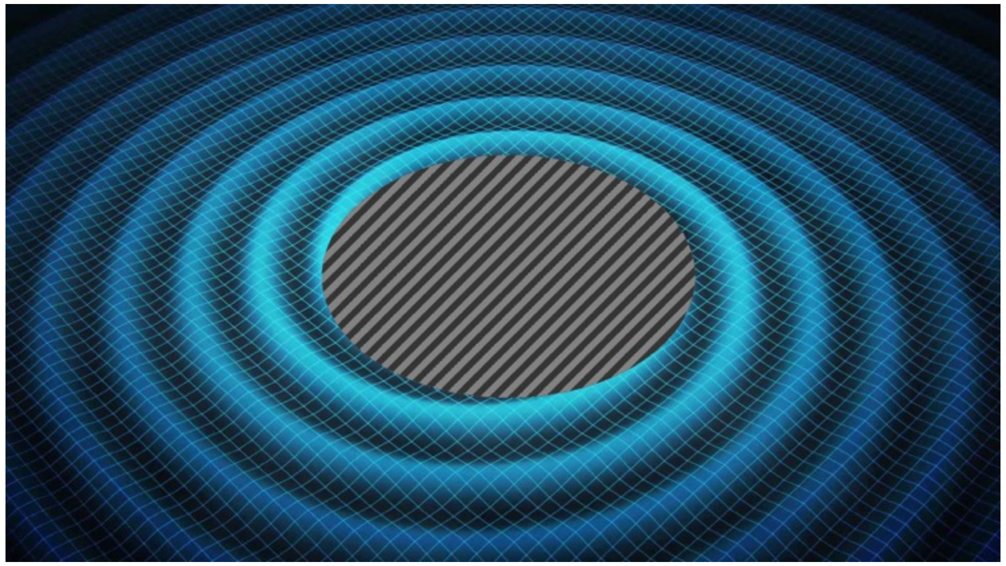

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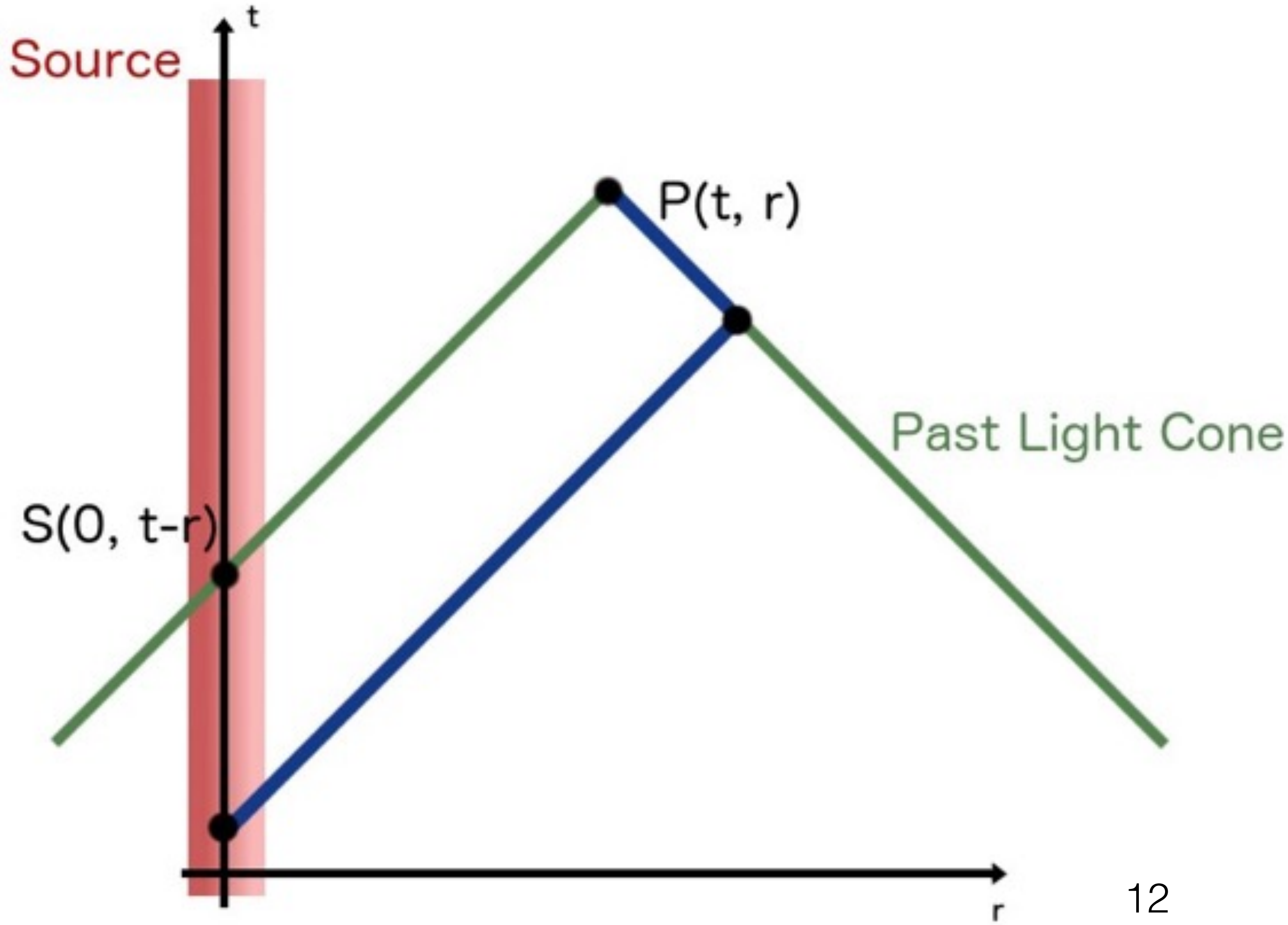

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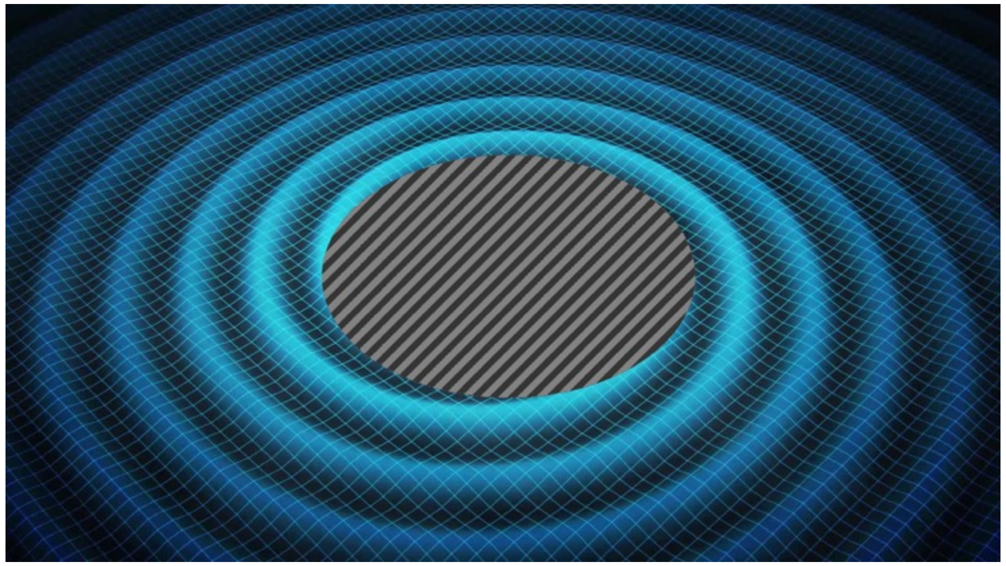

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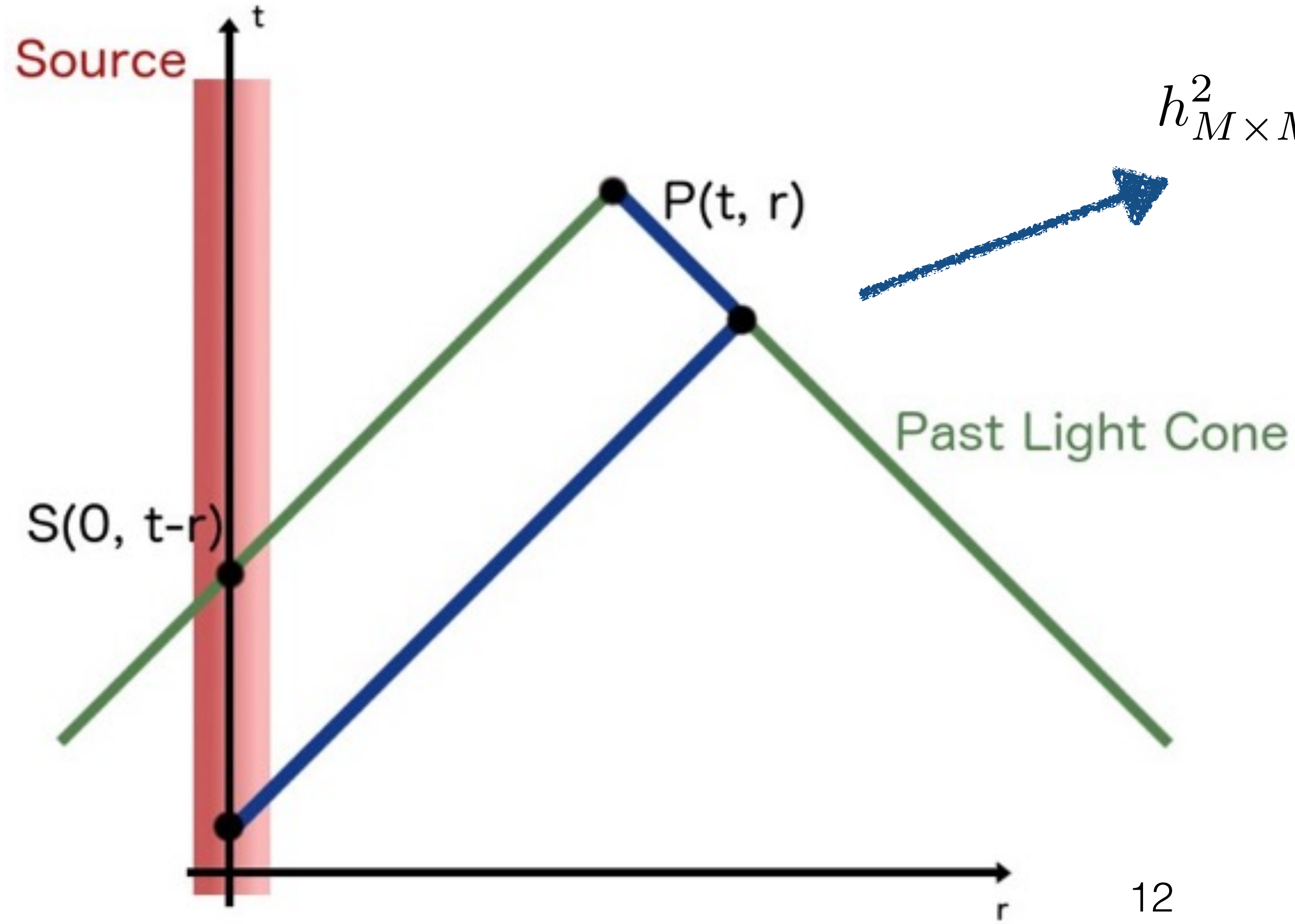

$$h^2_{M\times M_{ij}}(t,r) \sim M \int_0^\infty \mathrm{d}\tau M_{ij}(t-r-\tau)\mathcal{Q}(\tau)$$

III. The 4.5PN project

4.5PN project

Ultimate Goal: compute the flux up to 4.5PN

Done so far: compute all the 4.5PN contributions of the tails:

$$h^2_{M\times M_{ij}},\ h^3_{M\times M\times M_{ij}},\ h^4_{M\times M\times M\times M_{ij}}$$

arxiv:1607.07601

- Required new analytical formulae

‣ Required new analytical formulae

$$\mathrm{FP}_{B=0}\Box^{-1}\left[\hat{n}_L\left(\frac{r}{r_0}\right)^B r^{-k}\int_1^\infty \mathrm{d}y Q_m(y)F(t-ry)\right]$$
$$= -\hat{n}_L\int_1^\infty \mathrm{d}s F^{(k-2)}(t-rs)\left(Q_l(s)\int_1^s \mathrm{d}y Q_m^{(-k+2)}(y)P_l(y)+P_l(s)\int_s^\infty \mathrm{d}y Q_m^{(-k+2)}(y)Q_l(y)\right)$$

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$$\mathrm{FP}_{B=0}\Box^{-1}\left[\hat{n}_L\left(\frac{r}{r_0}\right)^B \ln\left(\frac{r}{r_0}\right)\frac{F(t-r)}{r^2}\right]$$
$$= -\frac{\hat{n}_L}{2r}\int_r^\infty \mathrm{d}s F(t-s)Q_m\left(\frac{s}{r}\right)\left[\ln\left(\frac{s^2-r^2}{4r_0^2}\right)+2H_l\right]$$

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‣ Implementing the algorithm into Mathematica

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Implementing the algorithm into Mathematica

➡ $h^2_{M\times M_{ij}}, \; h^3_{M\times M\times M_{ij}}, \; h^4_{M\times M\times M\times M_{ij}}$

Going to future null infinity

$$h^2_{M\times M_{ij}},\ h^3_{M\times M\times M_{ij}},\ h^4_{M\times M\times M\times M_{ij}}\ \text{but}\ h \underset{\substack{r\to\infty \\ t-r=\text{const}}}{\sim} \frac{\ln^k r}{r}$$

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- $$X^\mu = x^\mu + \xi^\mu(x) \qquad \begin{cases} \xi^0 = -2M\ln\left(\frac{r}{b_0}\right) \\ \xi^i = 0 \end{cases}$$

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- $$\begin{aligned} U_{ij}(T_R) &= M^{(2)}_{ij}(T_R) + \frac{GM}{c^3}\int_0^{+\infty} d\tau\, M^{(4)}_{ij}(T_R-\tau)\left[2\ln\left(\frac{c\tau}{2b_0}\right)+\frac{11}{6}\right] \\ &+ \frac{G^2M^2}{c^6}\int_0^{+\infty} d\tau\, M^{(5)}_{ij}(T_R-\tau)\left[2\ln^2\left(\frac{c\tau}{2b_0}\right)+\frac{11}{3}\ln\left(\frac{c\tau}{2b_0}\right) \right.\\ &\qquad \left. -\frac{214}{105}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{22050}\right] \\ &+ \frac{G^3M^3}{c^9}\int_0^{+\infty} d\tau\, M^{(6)}_{ij}(T_R-\tau)\left[\frac{4}{3}\ln^3\left(\frac{c\tau}{2b_0}\right)+\frac{11}{3}\ln^2\left(\frac{c\tau}{2b_0}\right) \right.\\ &\qquad +\frac{124627}{11025}\ln\left(\frac{c\tau}{2b_0}\right)-\frac{428}{105}\ln\left(\frac{c\tau}{2b_0}\right)\ln\left(\frac{c\tau}{2r_0}\right) \\ &\qquad \left. -\frac{1177}{315}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{129268}{33075}+\frac{428}{315}\pi^2\right] + \mathcal{O}\left(\frac{1}{c^{12}}\right) \end{aligned}$$

arxiv:1607.07601

$$\mathcal{F} = \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \left[a_l \left(U_L^{(1)} \right)^2 + \frac{b_l}{c^2} \left(V_L^{(1)} \right)^2 \right]$$

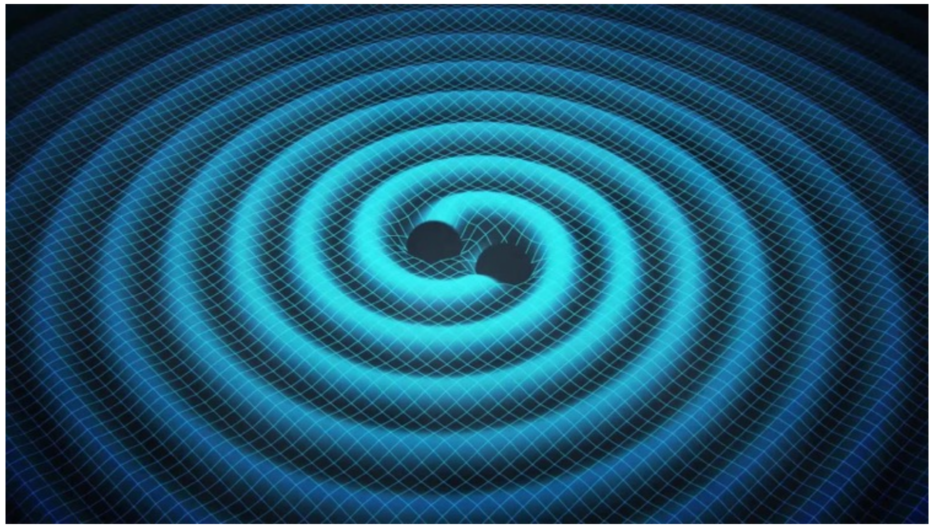
At 4.5PN, only non-local terms contribute in the flux

$$\mathcal{F}_{4.5\mathrm{PN}} = \mathcal{F}_{\mathrm{quadratic}} + \mathcal{F}_{\mathrm{quartic}}$$

Tails

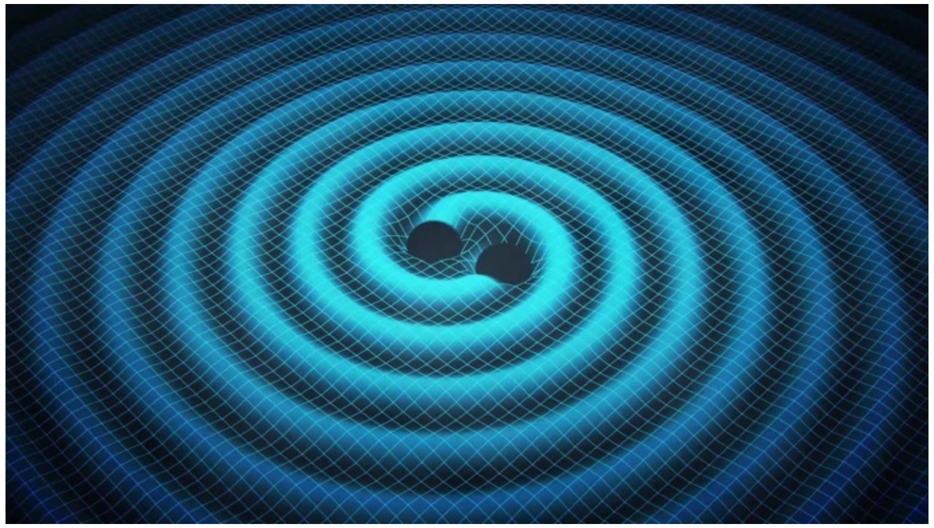
Tails-of-tails $\times$ tails

Tails-of-tails-of-tails



$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$\gamma = \frac{Gm}{rc^2}$$



$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

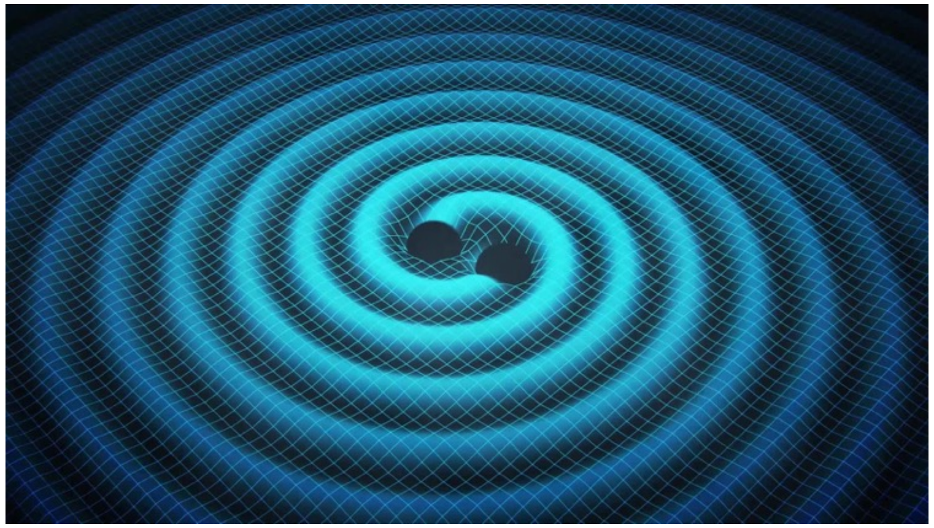
$$\gamma = \frac{Gm}{rc^2}$$

Tails

$$\mathcal{F}_{\rm quadratic} = \frac{32c^5}{5G}\nu^2\gamma^5\Big\{4\pi\gamma^{3/2} + \ldots + \left(\frac{9997778801}{106444800} - \frac{6848}{105}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{8058312817}{2661120} + \frac{287}{32}\pi^2 + \frac{572}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu - \frac{12433367}{13824}\nu^2 - \frac{1026257}{266112}\nu^3\right)\pi\gamma^{9/2} + \mathcal{O}\left(\frac{1}{c^{11}}\right)\Big\},$$

Tails-of-tails-of-tails and Tails-of-tails × tails

$$\mathcal{F}_{\rm quartic} = \frac{32c^5}{5G}\nu^2\gamma^5\Big\{\left(-\frac{467044}{3675} - \frac{3424}{105}\ln(16\gamma) + \frac{6848}{105}\ln\left(\frac{r}{r_0}\right) - \frac{6848}{105}\gamma_{\rm E}\right)\pi\gamma^{9/2} + \mathcal{O}\left(\frac{1}{c^{11}}\right)\Big\}.$$



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Tails

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Tails-of-tails-of-tails and Tails-of-tails × tails

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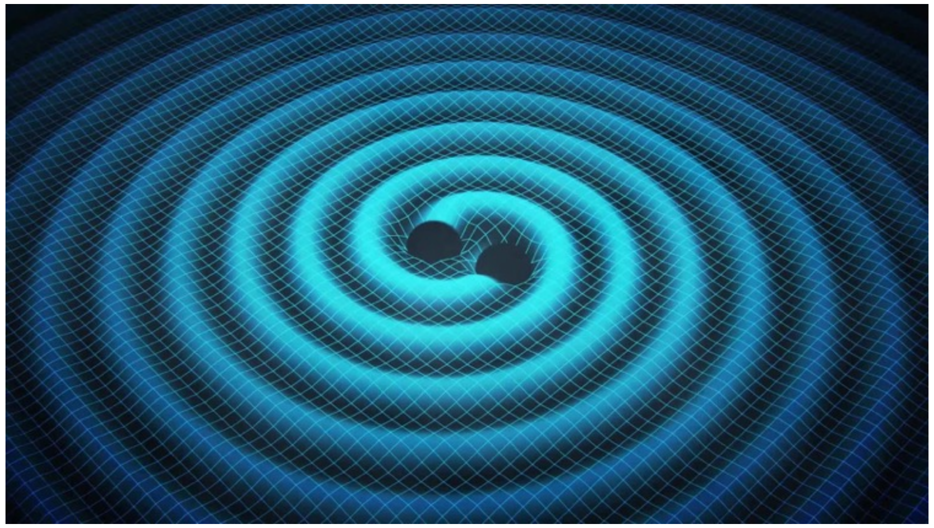
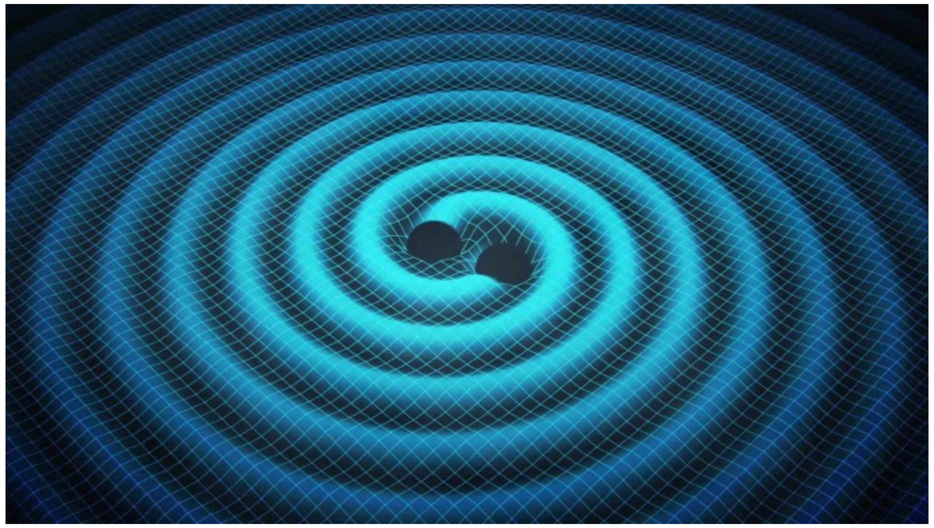


$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$x = \left(\frac{Gm\Omega}{c^3}\right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2}\right)$$

arxiv:1607.07601



$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

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$$\begin{aligned}
\mathcal{F}_{\text{total}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 \\
& + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right) \pi x^{5/2} + \Bigg[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E \\
& - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right) \pi x^{7/2} + (\text{unknown coefficients}) x^4 \\
& + \Bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_{\text{E}} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu \\
& - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Bigg) \pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}.
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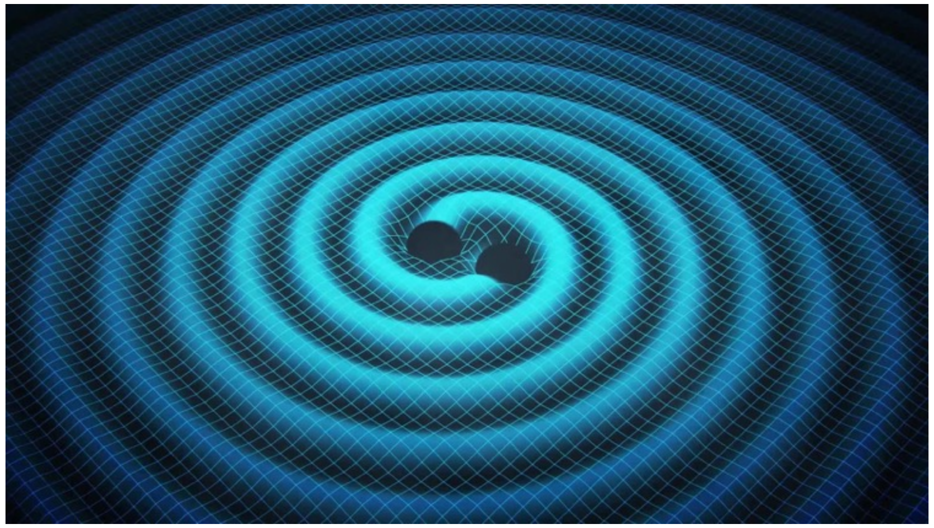

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Tanaka et al. gr-qc/9701050

arxiv:1607.07601

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Thank you !